14

Unsteady Flow

From olden times fluid had mostly been utilised mechanically for generating motive power, but recently it has been utilised for transmitting or automatically controlling power too. High-pressure fluid has to be used in these systems for high speed and good response. Consequently the issue of unsteady flow has become very important.

14.1 Vibration of liquid column in a U-tube

When the viscous frictional resistance is zero, by Newton's laws (Fig. 14.1),

$$\rho g(z_2 - z_1)A = -\rho A l \frac{\mathrm{d}v}{\mathrm{d}t} \tag{14.1}$$

$$g(z_2 - z_1) + l\frac{\mathrm{d}v}{\mathrm{d}t} = 0 \tag{14.2}$$

Moving the datum for height to the balanced state position, then

$$g(z_2 - z_1) = 2gz$$

Also,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 z}{\mathrm{d}t^2}$$

and from above,

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = -\frac{2g}{l}z \tag{14.3}$$

Therefore

$$z = C_1 \cos\sqrt{\frac{2g}{l}}t + C_2 \sin\sqrt{\frac{2g}{l}}t \tag{14.4}$$

Assuming the initial conditions are $t = 0$ and $z = z_0$, then $\mathrm{d}z/\mathrm{d}t = 0$, $C_1 = z_0$, $C_2 = 0$. Therefore

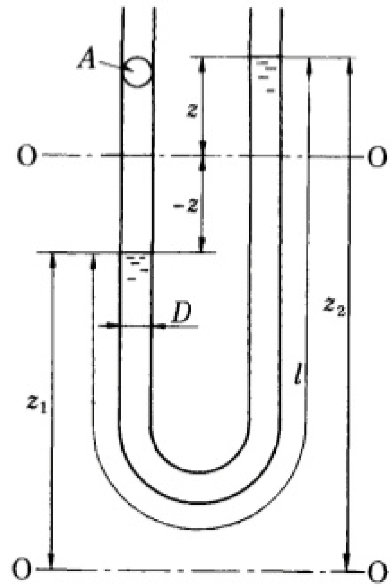

Fig. 14.1 Vibration of liquid column in a U-tube

$$z = z_0 \cos \sqrt{\frac{2g}{l}}\, t \tag{14.5}$$

This means that the liquid surface makes a singular vibration of cycle $T = 2\pi\sqrt{l/2g}$.

14.1.2 Laminar frictional resistance

In this case, with the viscous frictional resistance in eqn (14.1),

$$g(z_2 - z_1) + l\frac{dv}{dt} + \frac{32\nu v l}{D^2} = 0 \tag{14.6}$$

Substituting $2z = (z_2 - z_1)$ as above,

$$\frac{d^2z}{dt^2} + \frac{32\nu}{D^2}\frac{dz}{dt} + \frac{2g}{l}z = 0$$

$$\frac{d^2z}{dt^2} + 2\zeta\omega_n\frac{dz}{dt} + \omega_n^2 z = 0 \quad \text{where } \omega_n = \sqrt{\frac{2g}{l}} \text{ and } \zeta = \frac{16\nu}{D^2}\frac{1}{\omega_n} \tag{14.7}$$

The general solution of eqn (14.7) is as follows:

(a) when $\zeta < 1$

$$z = e^{-\zeta\omega_n t}\left[C_1 \sin\left(\omega_n\sqrt{1-\zeta^2 t}\right) + C_2 \cos\left(\omega_n\sqrt{1-\zeta^2 t}\right)\right] \tag{14.8}$$

Assume $z = z_0$ and $dz/dt = 0$ when $t = 0$. Then

$$\left.\begin{aligned} z &= z_0 e^{-\zeta\omega_n t}\left[\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)+\cos\left(\omega_n\sqrt{1-\zeta^2}t\right)\right] \\ &= \frac{z_0}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t+\phi\right) \\ \phi &= \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \end{aligned}\right\} \qquad (14.9)$$

(b) when $\zeta > 1$

$$\left.\begin{aligned} z &= z_0 e^{-\zeta\omega_n t}\left[\frac{\zeta}{\sqrt{\zeta^2-1}}\sinh\left(\omega_n\sqrt{\zeta^2-1}t\right)+\cosh\left(\omega_n\sqrt{\zeta^2-1}t\right)\right] \\ &= \frac{z_0}{\sqrt{\zeta^2-1}}e^{-\zeta\omega_n t}\sinh\left(\omega_n\sqrt{\zeta^2-1}t+\phi\right) \\ \phi &= \tanh^{-1}\left(\frac{\sqrt{\zeta^2-1}}{\zeta}\right) \end{aligned}\right\} \qquad (14.10)$$

Equations (14.9) and (14.10) can be plotted using the non-dimensional quantities of $\omega_n t$, z/z_0 as shown in Fig. 14.2. With large frictional resistance there is no oscillation but as it becomes smaller a damped oscillation occurs.

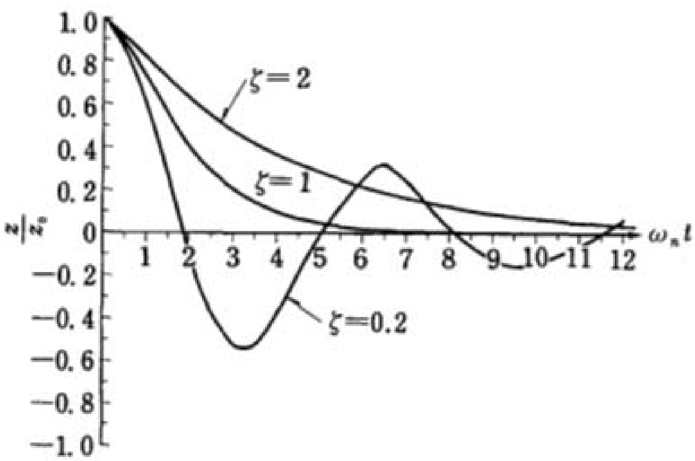

Fig. 14.2 Motion of liquid column with frictional resistance

14.2 Propagation of pressure in a pipe line

In the system shown in Fig. 14.3, a tank (capacity V) is connected to a pipe line (diameter D, section area A and length l). If the inlet pressure is suddenly

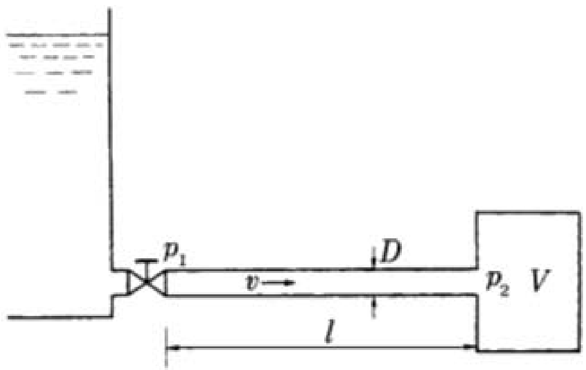

Fig. 14.3 System comprising pipe line and tank

changed (from 0 to p_1, say), it is desirable to know the response of the outlet pressure p_2. Assuming a pressure loss Δp due to tube friction, with instantaneous flow velocity v the equation of motion is

$$\rho A l = \frac{dv}{dt} = A(p_1 - p_2 - \Delta p) \tag{14.11}$$

If v is within the range of laminar flow, then

$$\Delta p = \frac{32\mu l}{D^2} v \tag{14.12}$$

Taking only the fluid compressibility β into account since the pipe is a rigid body,

$$dp_2 = \frac{1}{\beta}\frac{Av\,dt}{V} \tag{14.13}$$

Substituting eqns (14.12) and (14.13) into (14.11) gives

$$\frac{d^2p_2}{dt^2} + \frac{32\nu}{D^2}\frac{dp_2}{dt} + \frac{A}{\rho l \beta V}(p_2 - p_1) = 0$$

Now, writing

$$\omega_n = \sqrt{\frac{A}{\rho l \beta V}} \quad \zeta = \frac{16\nu}{D^2}\frac{1}{\omega_n} \quad p_2 - p_1 = p$$

then

$$\frac{d^2p}{dt^2} + 2\zeta\omega_n\frac{dp}{dt} + \omega_n^2 p = 0 \tag{14.14}$$

Since eqn (14.14) has the same form as eqn (14.7), the solution also has the same form as eqn (14.9) with the response tendency being similar to that shown in Fig. 14.2.

14.3 Transitional change in flow quantity in a pipe line

When the valve at the end of a pipe line of length l as shown in Fig. 14.4 is instantaneously opened, there is a time lapse before the flow reaches steady state. When the valve first opens, the whole of head H is used for accelerating the flow. As the velocity increases, however, the head used for acceleration decreases owing to the fluid friction loss h_1 and discharge energy h_2. Consequently, the effective head available to accelerate the liquid in the pipe becomes $\rho g(H - h_1 - h_2)$. So the equation of motion of the liquid in the pipe is as follows, putting A as the sectional area of the pipe,

$$\rho g A(H - h_1 - h_2) = \frac{\rho g A l}{g}\frac{\mathrm{d}v}{\mathrm{d}t} \tag{14.15}$$

giving

$$h_1 = \lambda\frac{l}{\mathrm{d}}\frac{v^2}{2g} = k\frac{v^2}{2g} \quad h_2 = \frac{v^2}{2g}$$

and

$$H - (k+1)\frac{v^2}{2g} = \frac{l}{g}\frac{\mathrm{d}v}{\mathrm{d}t} \tag{14.16}$$

Assume that velocity v becomes v_0 (terminal velocity) in the steady state ($\mathrm{d}v/\mathrm{d}t = 0$). Then

$$(k+1)v_0^2 = 2gh$$

$$k = \frac{2gh}{v_0^2} - 1$$

Substituting the value of k above into eqn (14.16),

$$H\left(1 - \frac{v^2}{v_0^2}\right) = \frac{l}{g}\frac{\mathrm{d}v}{\mathrm{d}t}$$

$$\mathrm{d}t = \frac{l}{gH}\frac{v_0^2}{v_0^2 - v^2}\mathrm{d}v$$

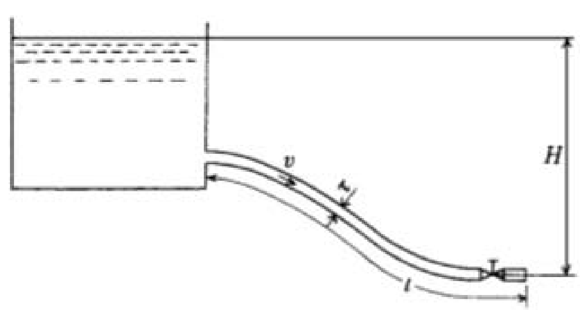

Fig. 14.4 Transient flow in a pipe

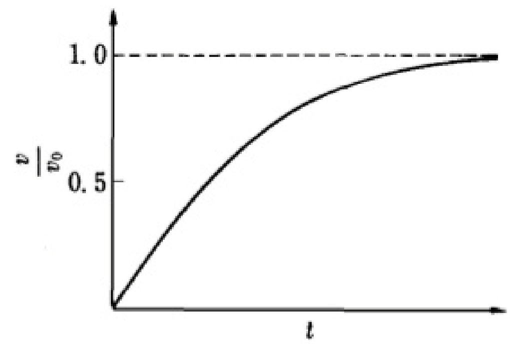

Fig. 14.5 Development of steady flow

or

$$t = \frac{lv_0}{2gH}\log\left(\frac{v_0 + v}{v_0 - v}\right) \tag{14.17}$$

Thus, time t for the flow to become steady is obtainable (Fig. 14.5).

Now, calculating the time from $v/v_0 = 0$ to $v/v_0 = 0.99$, the following equation can be obtained:

$$t = \frac{lv_0}{2gh}\log\left(\frac{1.99}{0.01}\right) = 2.646\frac{lv_0}{gh} \tag{14.18}$$

14.4 Velocity of pressure wave in a pipe line

The velocity of a pressure wave depends on the bulk modulus K (eqn (13.30)). The bulk modulus expresses the relationship between change of pressure on a fluid and the corresponding change in its volume. When a small volume V of fluid in a short length of rigid pipe experiences a pressure wave, the resulting reduction in volume dV_1 produces a reduction in length. If the pipe is elastic, however, it will experience radial expansion causing an increase in volume dV_2. This produces a further reduction in the length of volume V. Therefore, to the wave, the fluid appears more compressible, i.e. to have a lower bulk modulus. A modified bulk modulus K' is thus required which incorporates both effects.

From the definition equation (2.10),

$$-\frac{dp}{K} = \frac{dV_1}{V} \tag{14.19}$$

where the minus sign was introduced solely for the convenience of having positive values of K. Similarly, for positive K',

$$-\frac{dp}{K'} = \frac{dV_1 - dV_2}{V} \tag{14.20}$$

where the negative dV_2 indicates that, despite being a volume increase, it produces the equivalent effect of a volume reduction dV_1. Thus

$$\frac{1}{K'} = \frac{1}{K} + \frac{dV_2}{V\,dp} \tag{14.21}$$

If the elastic modulus (Young's modulus) of a pipe of inside diameter D and thickness b is E, the stress increase $d\sigma$ is

$$d\sigma = E\frac{dD}{D}$$

This hoop stress in the wall balances the internal pressure dp,

$$d\sigma = \frac{D}{2b}dp$$

Therefore

$$\frac{dD}{D} = \frac{D\,dp}{2bE}$$

Since $V = \pi D^2/4$ and $dV_2 = \pi D\,dD/2$ per unit length,

$$\frac{dV_2}{V} = 2\frac{dD}{D} = \frac{D\,dP}{bE} \tag{14.22}$$

Substituting eqn (14.22) into (14.21), then

$$\frac{1}{K'} = \frac{1}{K} + \frac{D}{bE}$$

or

$$K' = \frac{K}{1 + (D/b)(K/E)} \tag{14.23}$$

The sonic velocity a_0 in the fluid is, from eqn (13.30),

$$a_0 = \sqrt{K/\rho}$$

Therefore, the propagation velocity of the pressure wave in an elastic pipe is

$$a = \sqrt{\frac{K'}{\rho}} = \sqrt{\frac{K/\rho}{1 + (D/b)(K/E)}} = a_0\sqrt{\frac{1}{1 + (D/b)(K/E)}} \tag{14.24}$$

Since the values of D for steel, cast iron and concrete are respectively 206, 92.1 and 20.6 GPa, a is in the range 600–1200 m/s in an ordinary water pipe line.

14.5 Water hammer

Water flows in a pipe as shown in Fig. 14.6. If the valve at the end of the pipe is suddenly closed, the velocity of the fluid will abruptly decrease causing a mechanical impulse to the pipe due to a sudden increase in pressure of the fluid. Such a phenomenon is called water hammer. This phenomenon poses a

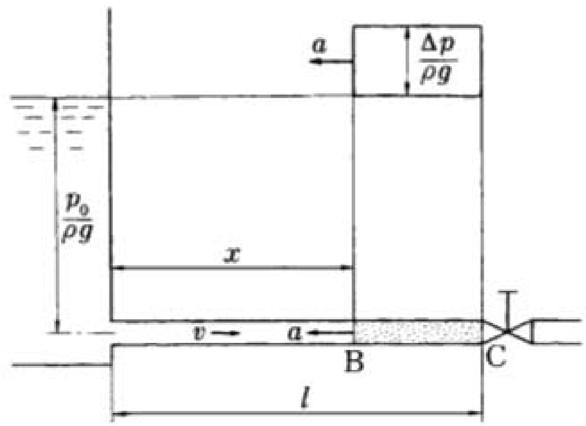

Fig. 14.6 Water hammer

very important problem in cases where, for example, a valve is closed to reduce the water flow in a hydraulic power station when the load on the water wheel is reduced. In general, water hammer is a phenomenon which is always possible whenever a valve is closed in a system where liquid is flowing.

14.5.1 Case of instantaneous valve closure

When the valve at pipe end C in Fig. 14.6 is instantaneously closed, the flow velocity v of the fluid in the pipe, and therefore also its momentum, becomes zero. Therefore, the pressure increases by dp. Since the following portions of fluid are also stopped one after another, dp propagates upstream. The propagation velocity of this pressure wave is expressed by eqn (14.24).

Given that an impulse is equal to the change of momentum,

$$dpA\frac{l}{a} = \rho Alv$$

or

$$dp = \rho va \tag{14.25}$$

When this pressure wave reaches the pipe inlet, the pressurised pipe begins to discharge backwards into the tank at velocity v. The pressure reverts to the original tank pressure p_0, and the pipe, too, begins to contract to its original state. The low pressure and pipe contraction proceed from the tank end towards the valve at velocity a with the fluid behind the wave flowing at velocity v. In time $2l/a$ from the valve closing, the wave reaches the valve. The pressure in the pipe has reverted to the original pressure, with the fluid in the pipe flowing at velocity v. Since the valve is closed, however, the velocity there must be zero. This requires a flow at velocity $-v$ to propagate from the valve. This outflow causes the pressure to fall by dp. This $-dp$ propagates

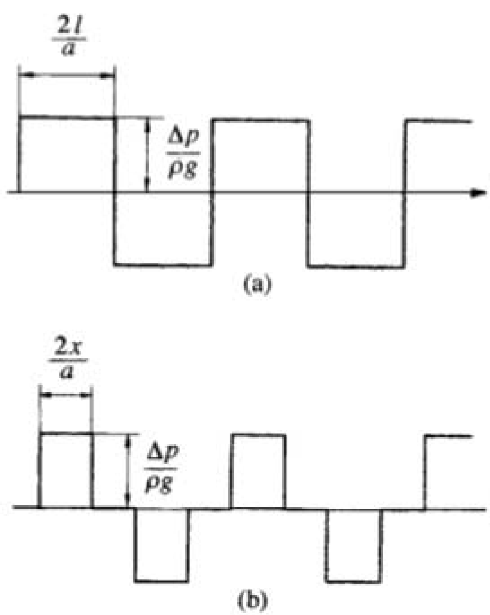

Fig. 14.7 Change in pressure due to water hammer, (a) at point C and (b) at point B in Fig. 14.6

upstream at velocity a. At time $3l/a$, from the valve closing, the liquid in the pipe is at rest with a uniform low pressure of $-\mathrm{d}p$. Then, once again, the fluid flows into the low pressure pipe from the tank at velocity v and pressure p. The wave propagates downstream at velocity a. When it reaches the valve, the pressure in the pipe has reverted to the original pressure and the velocity to its original value. In other words, at time $4l/a$ the pipe reverts to the state when the valve was originally closed. The changes in pressure at points C and B in Fig. 14.6 are as shown in Fig. 14.7(a) and (b) respectively. The pipe wall around the pressurised liquid also expands, so that the waves propagate at velocity a as shown in eqn (14.24).

14.5.2 Valve closure in time t_c

When the valve closing time t_c is less than time $2l/a$ for the wave round-trip of the pipe line, the maximum pressure increase when the valve is closed is equal to that in eqn (14.25).

When the valve closing time t_c is longer than time $2l/a$, it is called slow closing, to which Allievi's equation applies (named after L. Allievi (1856–1941), Italian hydraulics scholar). That is,

$$\frac{p_{max}}{p_0} = 1 + \tfrac{1}{2}\left(n^2 + n\sqrt{n^2 + 4}\right) \qquad (14.26)$$

Here, p_{max} is the highest pressure generated when the valve is closed, p_0 is the pressure in the pipe when the valve is open, v is the flow velocity when the valve is open, and $n = \rho l v/(p_0 t_c)$. This equation does not account for pipe friction and the valve is assumed to be uniformly closed.

In practice, however, there is pipe friction and valve leakage occurs. To obtain such changes in the flow velocity or pressure, either graphical analysis[1] or computer analysis (see Section 15.1) using the method of characteristics may be used.

14.6 Problems

1. As shown in Fig. 14.8, a liquid column of length 1.225 m in a U-shaped pipe is allowed to oscillate freely. Given that at $t = 0$, $z = z_0 = 0.4$ m and $dz/dt = 0$, obtain

   (a) the velocity of the liquid column when $z = 0.2$ m, and

   (b) the oscillation cycle time.

   Ignore frictional resistance.

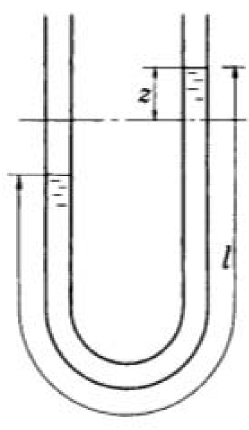

Fig. 14.8

2. Obtain the cycle time for the oscillation of liquid in a U-shaped tube whose arms are both oblique (Fig. 14.9). Ignore frictional resistance.

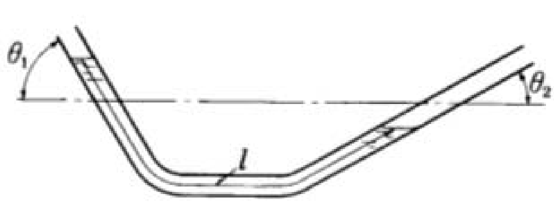

Fig. 14.9

[1] Parmakian, J., *Waterhammer Analysis*, (1963), 2nd edition, Dover, New York.

3. Oil of viscosity $\nu = 3 \times 10^{-5}\,\mathrm{m^2/s}$ extends over a 3 m length of a tube of diameter 2.5 cm, as shown in Fig. 14.8. Air pressure in one arm of the U-tube, which produces 40 cm of liquid column difference, is suddenly released causing the liquid column to oscillate. What is the maximum velocity of the liquid column if laminar frictional resistance occurs?

4. As shown in Fig. 14.10, a pipe line of diameter 2 m and length 400 m is connected to a tank of head 18 m. Find the time from the sudden opening of the valve for the exit velocity to reach 90% of the final velocity. Use a friction coefficient for the pipe of 0.03.

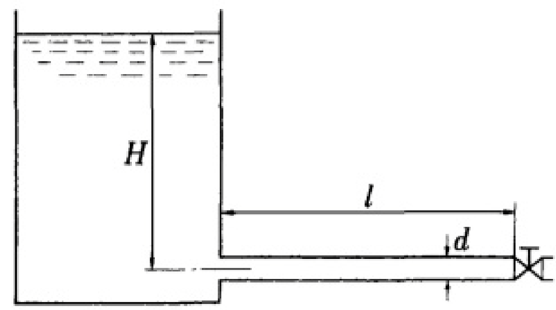

Fig. 14.10

5. Find the velocity of a pressure wave propagating in a water-filled steel pipe of inside diameter 2 cm and wall thickness 1 cm, if the bulk modulus $K = 2.1 \times 10^9\,\mathrm{Pa}$, density $\rho = 1000\,\mathrm{kg/m^3}$ and Young's modulus for steel $E = 2.1 \times 10^{11}\,\mathrm{Pa}$.

6. Water flows at a velocity of 3 m/s in the steel pipe in Problem 5, of length 1000 m. Obtain the increase in pressure when the valve is shut instantaneously.

7. The steady-state pressure of water flowing in the pipe line in Problem 6, at a velocity of 3 m/s, is $5 \times 10^5\,\mathrm{Pa}$. What is the maximum pressure reached when the valve is shut in 5 seconds?