الجامعة التكنولوجية

قسم الهندسة الكيمياوية

المرحلة الرابعة

السيطرة على العمليات

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Introduction to process control

Water Heater

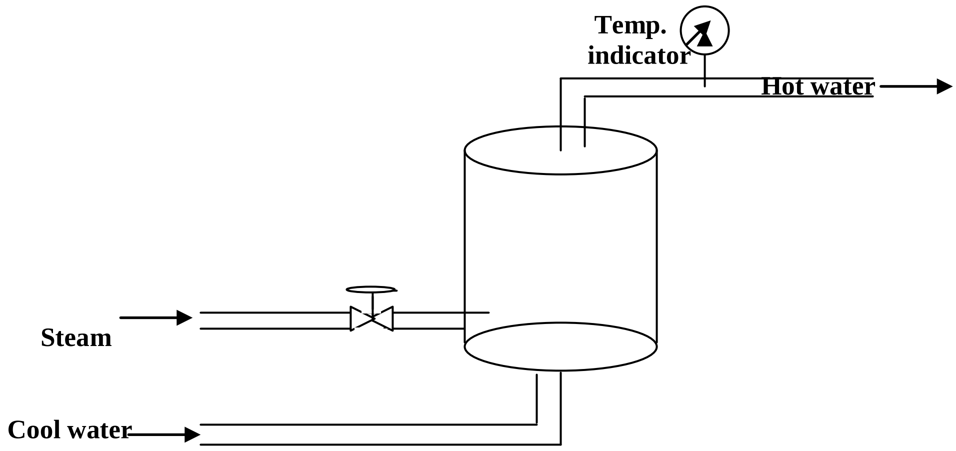


Figure (1) Open loop system

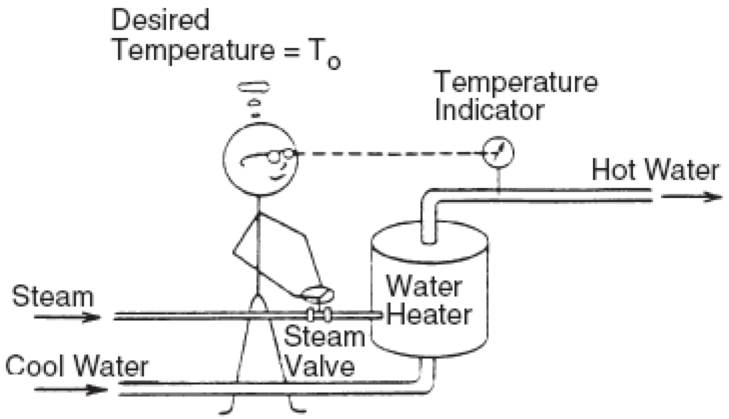


Figure (2) Manual Control system

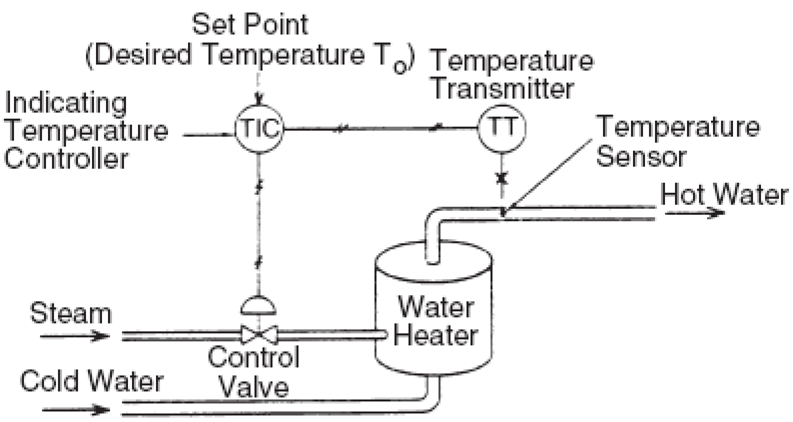


Figure (3) Automatic Control system (Closed Loop)

Control System Objectives

- Economic Incentive
- Safety
- Equipment Protection
- Reduce variability
- Increase efficiency
- Ensure the stability of a process
- Elimination of routine

Definitions:

System: It is a combination of components that act together and perform a certain objective.

Plant: It is the machine of which a particular quantity or condition is to be controlled.

Process: Is defined as the changing or refining of raw materials that pass through or remain in a liquid, gaseous, or slurry state to create end products.

Control: In process industries refers to the regulation of all aspects of the process. Precise control of level, pH, oxygen, foam, nutrient, temperature, pressure and flow is important in many process applications.

Sensor: A measuring instrument, the most common measurements are of flow (F), temperature (T), pressure (P), level (L), pH and composition (A, for analyzer). The sensor will detect the value of the measured variable as a function of time.

Set point: The value at which the controlled parameter is to be maintained.

Controller: A device which receives a measurement of the process variable, compares with a set point representing the desired control point, and adjusts its output to minimize the error between the measurement and the set point.

Error Signal: The signal resulting from the difference between the set point reference signal and the process variable feedback signal in a controller.

Feedback Control: A type of control whereby the controller receives a feedback signal representing the condition of the controlled process variable, compares it to the set point, and adjusts the controller output accordingly.

Steady-State: The condition when all process properties are constant with time, transient responses having died out.

Transmitter: A device that converts a process measurement (pressure, flow, level, temperature, etc.) into an electrical or pneumatic signal suitable for use by an indicating or control system.

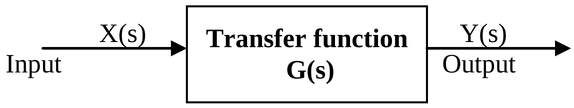
Controlled variable: Process output which is to be maintained at a desired value by adjustment of a process input.

Manipulated variable: Process input which is adjusted to maintain the controlled output at set point.

Disturbance: A process input (other than the manipulated parameter) which affects the controlled parameter.

Process Time Constant(τ): Amount of time counted from the moment the variable starts to respond that it takes the process variable to reach 63.2% of its total change.

Block diagram: It is relationship between the input and the output of the system. It is easier to visualize the control system in terms of a block diagram.

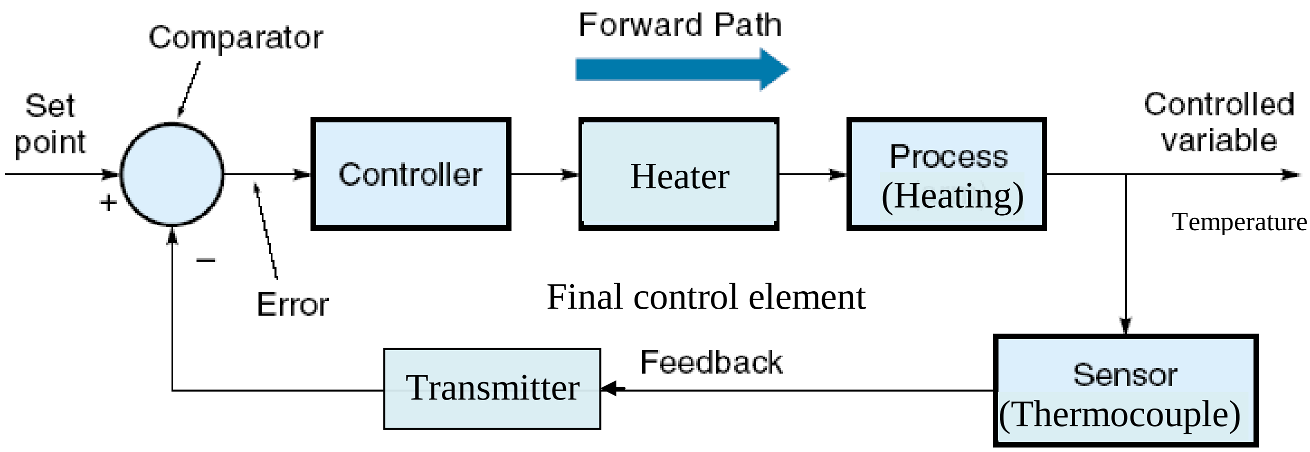


Block diagram

Transfer Function: it is the ratio of the Laplace transform of output (response function) to the Laplace transform of the input (driving force) under assumption that all initial conditions are zero unless that given another value.

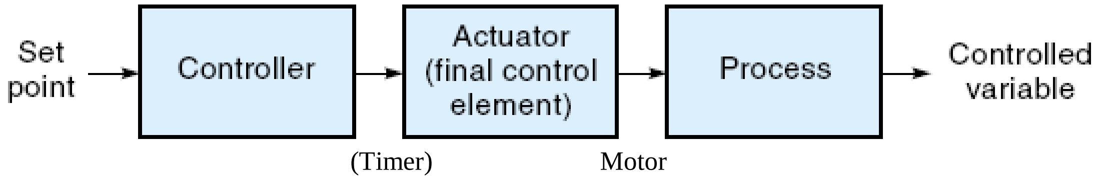
e.g. the transfer function of the above block diagram is G (s) = Y(s)/X(s)

Closed-loop control system: It is a feedback control system which the output signals has a direct effect upon the control action.



Advantage: more accurate than the open-loop control system.
Disadvantages: (1) Complex and expensive
(2) The stability is the major problem in closed-loop control system

Open-loop control system: It is a control system in which the output has no effect upon the control action. (The output is neither measured nor fed back for comparison with the input).



Advantages:
(1) Simple construction and ease of maintenance.
(2) Less expensive than closed-loop control system.
(3) There is no stability problem.

Disadvantages:
(1) Disturbance and change in calibration cause errors; and output may be different from what is desired.
(2) To maintain the required quality in the output, recalibration is necessary from time to time

<u>Note</u>: any control system which operates on a time basis is open-loop control system, e.g. washing machine, traffic light …etc.

Laplace Transforms

2.1 Introduction

Laplace transform techniques provide powerful tools in numerous fields of technology such as Control Theory where knowledge of the system transfer function is essential and where the Laplace transform comes into its own.

Definition

The Laplace transform of an expression $f(t)$ is denoted by $L\{f(t)\}$ and is defined as the semi-infinite integral: $L\{f(t)\}=\int_{t=0}^{\infty} f(t)e^{-st}dt$.

The parameter s is assumed to be positive and large enough to ensure that the integral converge. In more advanced applications s may be complex and in such cases the real part of s must be positive and large enough to ensure convergence.

In determining the transform of an expression, you will appreciate that the limits of the integral are substituted for t, so that the result will be an expression in s. Therefore: $\boxed{L\{f(t)\}=\int_{t=0}^{\infty} f(t)e^{-st}dt = F(s)}$

2.2 Simple Transforms

Example: Find the Laplace transform of $f(t)=1$

Solution: $L\{1\}=\int_{0}^{\infty} 1e^{-st}dt=-\left[\frac{e^{-st}}{s}\right]_0^{\infty}=-(\frac{e^{-s\infty}}{s}-\frac{e^{-s0}}{s})=-(\frac{0}{s}-\frac{1}{s})=\frac{1}{s}$

Example: Find the Laplace transform of $f(t)=a$, where a is a constant.

Solution: $L\{a\}=\int_{0}^{\infty} ae^{-st}dt=a\left[\frac{-e^{-st}}{s}\right]_0^{\infty}=a(\frac{-e^{-s\infty}}{s}+\frac{e^{-s0}}{s})=a(\frac{0}{s}+\frac{1}{s})=\frac{a}{s}$

Example: Find the Laplace Transform of $f(t)=t$

Solution: $\text{L}\{t\}=\int_0^{\infty} te^{-st}dt$

Use integration by parts with

$$\int_0^{\infty} u\cdot dv = u\cdot v\Big|_0^{\infty}-\int_0^{\infty} v\cdot du$$

$$u=t \Rightarrow du=dt$$

$$dv=e^{-st}dt \Rightarrow v=\frac{-e^{-st}}{s}$$

$$L[t]=\int_0^\infty te^{-st}dt=-t\cdot\frac{1}{s}e^{-st}\Big|_0^\infty-\int_0^\infty\left(-\frac{1}{s}e^{-st}\right)dt=t\cdot\frac{1}{s}e^{-st}\Big|_0^\infty+\frac{1}{s}\int_0^\infty e^{-st}dt$$

$$=(0-0)+\frac{1}{s}\left[-\frac{1}{s}e^{-st}\right]_0^\infty=\frac{e^{-st}}{s^2}=\frac{1}{s^2}(1-0)=\frac{1}{s^2}$$

$$\Rightarrow L\{t\}=\frac{1}{s^2}$$

Example: Find the Laplace Transform of $f(t)=t^2$

Solution: $\mathrm{L}\{t^2\}=\int_0^\infty t^2e^{-st}dt$

use integration by parts with

$$\int_0^\infty u\cdot dv=u\cdot v\Big|_0^\infty-\int_0^\infty v\cdot du$$

$$u=t^2\Rightarrow du=2tdt$$

$$dv=e^{-st}dt\Rightarrow v=\frac{e^{-st}}{-s}$$

$$\mathrm{L}\{t^2\}=\int_0^\infty t^2e^{-st}dt=\frac{t^2e^{-st}}{-s}\Big|_0^\infty-\left[-\int_0^\infty\frac{2te^{-st}}{s}dt\right]=\int_0^\infty\frac{2te^{-st}}{s}dt$$

$$\int_0^\infty u\cdot dv=u\cdot v\Big|_0^\infty-\int_0^\infty v\cdot du$$

$$u=2t\Rightarrow du=2dt$$

$$dv=\frac{e^{-st}}{s}dt\Rightarrow v=\frac{-e^{-st}}{s^s}$$

$$\mathrm{L}\{t^2\}=\int_0^\infty\frac{2te^{-st}}{s}dt=\frac{-2te^{-st}}{s^2}\Big|_0^\infty-\left[-\int_0^\infty\frac{2e^{-st}}{s^2}dt\right]=2\int_0^\infty\frac{e^{-st}}{s^2}dt=\frac{-2e^{-st}}{s^3}\Big|_0^\infty=-\frac{2}{s^3}(0-1)=\frac{2}{s^3}$$

$$\Rightarrow L\{t^2\}=\frac{2}{s^3}$$

Example: Find the Laplace Transform of $f(t)=e^{at}$, where a is a constant.

Solution:

$$L\{e^{at}\}=\int_0^\infty e^{at}e^{-st}dt=\int_0^\infty e^{at-st}dt=\int_0^\infty e^{-t(s-a)}dt=\left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_0^\infty=-\frac{1}{(s-a)}\left[e^{-t(s-a)}\right]_0^\infty=-\frac{1}{(s-a)}\{0-1\}$$

$$\Rightarrow \boxed{L\{e^{at}\}=\frac{1}{(s-a)}}$$

Similarity

$$\boxed{L\{e^{-at}\}=\frac{1}{(s+a)}}$$

Example: Find the Laplace Transform of $f(t) = \sin(at)$

Solution:

$$L\{\sin(at)\} = \int_0^\infty \sin(at) e^{-st} = \int_0^\infty \frac{e^{iat} - e^{-iat}}{2i} e^{-st} dt = \frac{1}{2i}\int_0^\infty e^{(ia-s)t} - e^{-(ia+s)t} dt$$

$$= \left(\frac{1}{ia-s} e^{(ia-s)t} \Big|_0^\infty - \frac{-1}{ia+s} e^{-(ia+s)t} \Big|_0^\infty \right) = \frac{1}{2i}\left[\frac{0-1}{ia-s} + \frac{0-1}{ia+s} \right] = \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia} \right]$$

$$= \frac{1}{2i}\left[\frac{(s+ia)-(s-ia)}{(s-ia)(s+ia)} \right] = \frac{1}{2i}\left[\frac{2ia}{s^2+a^2} \right] = \frac{a}{s^2+a^2}$$

$$\boxed{L\{\sin(at)\} = \frac{a}{s^2+a^2}}$$

Also

$$\boxed{L\{\cos(at)\} = \frac{s}{s^2+a^2}}$$

Example: Find the Laplace Transform of $f(t) = \sinh(at)$

Solution: $\sinh(at) = \dfrac{e^{at} - e^{-at}}{2}$

$$L\{\sinh(at)\} = L\left\{ \frac{e^{at} - e^{-at}}{2} \right\} = \frac{1}{2} L\{e^{at} - e^{-at}\} = \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a} \right) = \frac{1}{2}\left(\frac{(s+a)-(s-a)}{(s-a)(s+a)} \right)$$

$$= \frac{1}{2}\left(\frac{2a}{(s^2 - as + as + a^2)} \right) = \frac{a}{s^2 - a^2}$$

$$\boxed{L\{\sinh(at)\} = \frac{a}{s^2-a^2}}$$

Also

$$\boxed{L\{\cosh(at)\} = \frac{s}{s^2-a^2}}$$

In practice we do not usually need to integrate to find Laplace transforms, instead we use a table, which allow us to read off most of the transforms we need.

Function $f(t)$	**Transform** $F(s)$	**Valid for …**
1	$\frac{1}{s}$	$s > 0$
a	$\frac{a}{s}$	$s > 0$
t	$\frac{1}{s^2}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	n = positive integer
$\sin at$	$\frac{a}{s^2+a^2}$	

$\cos at$	$\frac{s}{s^2+a^2}$	
$\sinh at$	$\frac{a}{s^2-a^2}$	
$\cosh at$	$\frac{s}{s^2-a^2}$	
e^{-at}	$\frac{1}{s+a}$	
te^{-at}	$\frac{1}{(s+a)^2}$	
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$	
$e^{-at}\sin wt$	$\frac{w}{(s+a)^2+w^2}$	
$e^{-at}\cos wt$	$\frac{s+a}{(s+a)^2+w^2}$	

2.3 Rules of Laplace transform

The Laplace transform is a linear transform by which is meant that:

1. The transform of a sum (or difference) of expressions is the sum (or difference) of the individual transforms. That is

$$\boxed{L\{f(t)\pm g(t)\}=L\{f(t)\}\pm L\{g(t)\}}.$$

2. The transform of an expression that is multiplied by a constant is the constant multiplied by the transform of the expression. That is

$$\boxed{L\{kf(t)\}=kL\{f(t)\}}.$$

Example: Determine the Laplace transform of $2e^{-t}+t$.

Solution: $L\{2e^{-t}+t\}=2L\{e^{-t}\}+L\{t\}=\frac{2}{s+1}+\frac{1}{s^2}=\frac{2s^2+(s+1)}{s^2(s+1)}=\frac{2s^2+s+1}{(s^3+s^2)}$

Example: Determine the Laplace transform of $3t^3+\sin t$.

Solution: $L\{3t^3+\sin t\}=3L\{t^3\}+L\{\sin t\}=3\times\frac{3!}{s^{3+1}}+\frac{1}{s^2+1^2}=\frac{18}{s^4}+\frac{1}{s^2+1}$

$$=\frac{18(s^2+1)+1(s^4)}{s^4(s^2+1)}=\frac{18(s^2+1)+s^4}{s^4(s^2+1)}=\frac{18s^2+18+s^4}{s^4(s^2+1)}$$

2.4 Theorems of Laplace transform

There are three important and useful theorems that enable us to deal with rather more complicated expressions.

<u>Theorem 1: The first shift theorem</u>

The first shift theorem states that if $L\{f(t)\}=F(s)$ then $\boxed{L\{e^{-at}f(t)\}=F(s+a)}$

$$L\{e^{-at} f(t)\} = \int_{t=0}^{\infty} e^{-at} f(t) e^{-st} dt = \int_{t=0}^{\infty} f(t) e^{-(s+a)t} dt = F(s+a)$$

We know that $L\{e^{-at} f(t)\} = F(s+a)$ and we know that $L\{f(t)\} = F(s)$ therefore the transform $L\{e^{-at} f(t)\}$ is thus the same as $L\{f(t)\}$ with s everywhere in the result replaced by $(s+a)$.

Example: find $L\{e^{-3t}\sin 2t\}$.

Solution: We know that $L\{e^{-at}\} = \dfrac{1}{s+a}$ and $L\{\sin 2t\} = \dfrac{2}{s^2+4}$ We have $a = 3$, therefore

$$L\{e^{-3t}\sin 2t\} = \frac{2}{(s+3)^2+4} = \frac{2}{s^2+3s+3s+9+4} = \frac{2}{s^2+6s+13}$$

Example: Determine the Laplace transform of $e^{3t}(t^2+4)$.

Solution: We know that $L\{t^2+4\} = \dfrac{2}{s^3} + \dfrac{4}{s}$ also that $L\{e^{3t}\} = \dfrac{1}{s-3}$ Therefore

$$L\{e^{3t}(t^2+4)\} = \frac{2}{(s-3)^3} + \frac{4}{(s-3)} = \frac{2+4(s-3)^2}{(s-3)^3} = \frac{2+4(s-3)(s-3)}{(s-3)^3}.$$

$$= \frac{2+4(s^2-6s+9)}{(s-3)^3} = \frac{2+4s^2-24s+36}{(s-3)^3} = \frac{4s^2-24s+38}{(s-3)^3}$$

<u>Theorem 2: Multiplying by t and t</u>^{n}

If $L\{f(t)\} = F(s)$ then $\boxed{L\{tf(t)\} = -F'(s)}$

$$L\{tf(t)\} = \int_{t=0}^{\infty} tf(t) e^{-st} dt = \int_{t=0}^{\infty} f(t)\left(-\frac{de^{-st}}{ds}\right) dt = -\frac{d}{ds}\int_{t=0}^{\infty} f(t) e^{-st} dt = -F'(s).$$

In general if $L\{f(t)\} = F(s)$, then $\boxed{L\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n}\{F(s)\}}$

Example: find $L\{t\sin 2t\}$.

Solution: From $L\{tf(t)\} = -F'(s)$ therefore $L\{t\sin 2t\} = -\dfrac{d}{ds}\left(\dfrac{2}{s^2+4}\right)$.

NB: To find $\dfrac{d}{ds}\left(\dfrac{2}{s^2+4}\right)$ use quotient rule for differentiation:

$$z = \left(\frac{2}{s^2+4}\right), z = \frac{a}{b}, a = 2, b = s^2+4, \frac{da}{ds} = 0, \frac{db}{ds} = 2s$$

$$\frac{dz}{ds} = \frac{b\dfrac{da}{ds} - a\dfrac{db}{ds}}{b^2} = \frac{(s^2+4)(0)-(2)(2s)}{(s^2+4)^2} = \frac{-4s}{(s^2+4)^2}$$

$$L\{t\sin 2t\} = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = \frac{4s}{(s^2+4)^2}$$

Example: Determine the Laplace transform of $t^2 \sin t$,

Solution: We know that $L\{\sin t\} = \frac{1}{s^2+1}$, therefore we can work out from

$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{F(s)\}$ that

$$L\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2}\{F(s)\} = \frac{d^2}{ds^2}\{F(s)\} = \frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right)$$

Find the first derivative:

$$z = \frac{1}{s^2+1}, z = \frac{a}{b}, a = 1, b = s^2+1, \frac{da}{ds} = 0, \frac{db}{ds} = 2s$$

$$\frac{dz}{ds} = \frac{b\frac{da}{ds} - a\frac{db}{ds}}{b^2} = \frac{(s^2+1)(0)-(1)(2s)}{(s^2+1)^2} = \frac{-2s}{(s^2+1)^2}$$

Differentiate again.

$$z = \frac{a}{b} = \frac{-2s}{(s^2+1)^2}, a = -2s, b = (s^2+1)^2 = (s^2+1)(s^2+1) = s^4+2s^2+2,$$

$$\frac{da}{ds} = -2, \frac{db}{ds} = 4s^3+4s,$$

$$\frac{dz}{ds} = \frac{b\frac{da}{ds} - a\frac{db}{ds}}{b^2} = \frac{(s^2+1)^2(-2)-(-2s)(4s^3+4s)}{((s^2+1)^2)^2} = \frac{-2((s^2+1)^2)+(8s^4+8s^2)}{(s^2+1)^2(s^2+1)^2}$$

$$= \frac{-2(s^4+2s^2+2)+(8s^4+8s^2)}{(s^2+1)^2(s^2+1)^2} = \frac{(-2s^4-4s^2-4)+(8s^4+8s^2)}{(s^2+1)^2(s^2+1)^2} = \frac{(6s^4+4s^2-4)}{(s^2+1)^2(s^2+1)^2}$$

Therefore

$$L\{t^2 \sin t\} = \frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right) = \frac{(6s^4+4s^2-4)}{(s^2+1)^2(s^2+1)^2}.$$

Theorem 3: Convolution Theorem:

If $L\{f(t)\} = F(s)$ and $L\{g(t)\} = G(s)$ then the convolution of $f(t)$ and $g(t)$ is denoted by $(f * g)(t)$, is defined by

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

And the Laplace transform of the convolution of two functions is the product of the separate Laplace transforms:

$$L\{(f*g)(t)\} = F(s)G(s)$$

An equivalent identity is

$$L^{-1}\{F(s)G(s)\} = L^{-1}\{F(s)\} * L^{-1}\{G(s)\}$$

Example: Find $t^2 * 2t$

Solution:

$$t^2 * 2t = \int_0^t \tau^2 2(t-\tau)d\tau = \int_0^t (2t\tau^2 - 2\tau^3)d\tau = \left(2t\frac{\tau^3}{3} - 2\frac{\tau^4}{4}\right)\Bigg|_0^t = \frac{2t^4}{3} - \frac{t^4}{2} = \frac{t^4}{6}$$

Example: *Find* $e^t * t$

Solution:

$$e^t * t = \int_0^t e^{t-\tau}(\tau)d\tau = e^t\int_0^t \left(\tau e^{-\tau}\right)d\tau = e^t\left(-\tau e^{-\tau} - e^{-\tau}\right)\Big|_0^t$$

$$= e^t\left[\left(-te^{-t} - e^{-t}\right) - (0-1)\right] = -e^t e^{-t}(t+1) + e^t \quad = -t - 1 + e^t = e^t - t - 1$$

Example: find Laplace transform of $e^t\cos(t)$ by convolution theorem

$$f(t) = e^t \qquad g(t) = \cos(t)$$

$$L\{f*g\} = L\{f(t)\}L\{g(t)\} = L\{e^t\}L\{\cos(t)\}$$

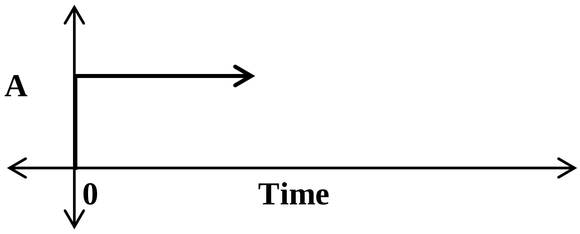
$$= \left(\frac{1}{s-1}\right)\left(\frac{s}{s^2+1^2}\right) = \frac{s}{(s-1)(s^2+1)}$$

2.5 Special Laplace Transform Functions

1- Step function

$$f(t) = \begin{cases} 0 & t < 0 \\ A & t \geq 0 \end{cases}$$

$$f(s) = \frac{A}{s}$$



If A=1 the change is called **unit step change**

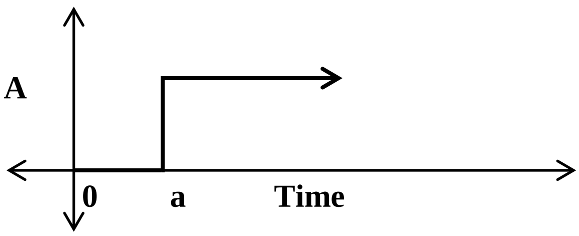
$$f(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

$$f(s) = \frac{1}{s}$$

Step function with Time Delay

$$f(t) = \begin{cases} 0 & t < a \\ A & t \geq a \end{cases}$$

$$f(s) = \frac{A}{s}e^{-as}$$

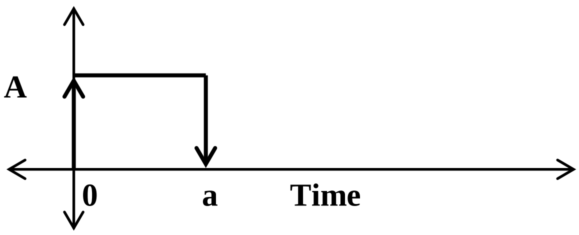


2. Pulse function

$$f(t) = \begin{cases} 0 & t < 0 \\ A & 0 \leq t \leq a \\ 0 & t \geq a \end{cases}$$

$$f(s) = \frac{A}{s} - \frac{A}{s}e^{-as}$$

$$= \frac{A}{s}(1 - e^{-as})$$



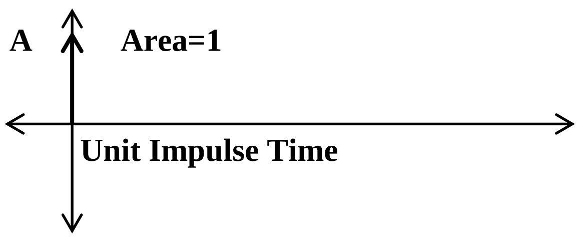
$$L\{f(t)\} = \int_0^a Ae^{-st} + \int_a^\infty 0e^{-st} = -\frac{At}{s}e^{-st}\Big|_0^a = -\frac{A}{s}(e^{-sa} - e^{-s0}) = \frac{A}{s}(1 - e^{-sa})$$

if A=1➔unit Pulse (Impulse)

3. Impulse function

$$f(t) = \begin{cases} 0 & t < 0 \\ A & 0 \le t \le \delta t \\ 0 & t \ge \delta t \end{cases}$$

$$f(s) = area = A \times \delta t$$



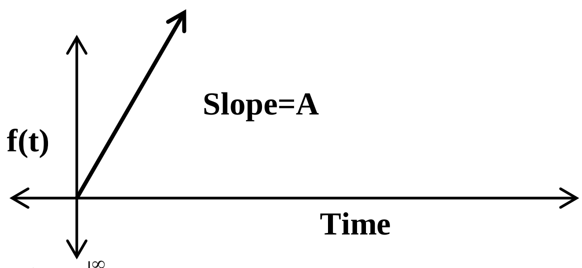
This function is represented by δ(t). The *unit impulse* function is a special case of the pulse function with zero width (tw →0) and unit pulse area (so a = 1/tw). Taking the limit:

$$L\{\delta(t)\} = \lim_{t_w \to 0} \frac{1}{t_w s}[1 - e^{-st_w}] = \lim_{t_w \to 0} \frac{1}{s}[-e^{-st_w}] = 1$$

4. Ramp function

$$f(t) = \begin{cases} 0 & t < 0 \\ At & t \ge 0 \end{cases}$$

$$f(s) = \frac{A}{s^2}$$



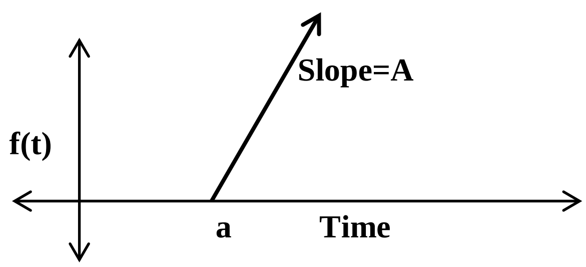
$$L\{At\} = \int_0^\infty Ate^{-st} = -\frac{At}{s}e^{-st}\Big|_0^\infty + \int \frac{A}{s}e^{-st}dt = -\frac{At}{s}e^{-st}\Big|_0^\infty + \frac{A}{s}\frac{-1}{s}e^{-st}\Big|_0^\infty$$

$$= \frac{-A}{s}(\infty e^{-\infty} - 0e^{-0}) - \frac{A}{s^2}(e^{-\infty} - e^{-0}) = \frac{A}{s^2}$$

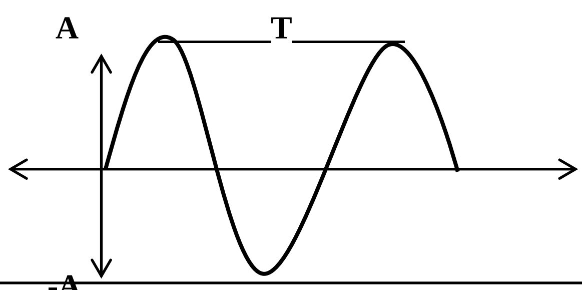
Ramp function with time delay

$$f(t) = \begin{cases} 0 & t < a \\ At & t \ge a \end{cases}$$

$$f(s) = \frac{A}{s^2}e^{-as}$$

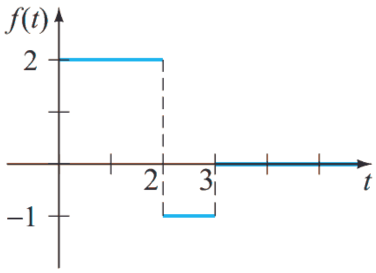


5. Sine function



$$f(t)=\begin{cases}0 & t<a\\ A\sin wt & t\geq a\end{cases}$$
$$f(s)=\frac{A\omega}{s^2+\omega^2}$$
$$\omega=2\pi f$$
$$T=\frac{1}{f}$$

Example: Find the Laplace transform for



Solution:

1. At t=0 the function looks like the very basic unit step function. But unit function knows only about 0 and 1, here we have f(t)=2. That means we have to use 2u(t).
2. Then in time t=2 its value changes from 2 to −1 (i.e. 3 down at t=2) which means we have to add −3u(t−2).
3. Finally the value at t=3 jumps 1 higher, which brings member u(t−3).

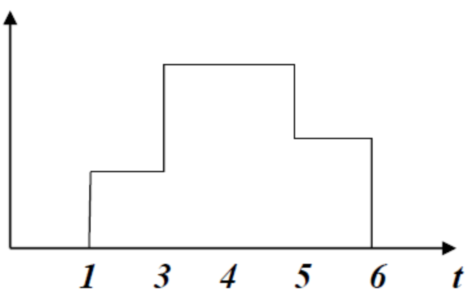
f(t)=2u(t)−3u(t−2)+u(t−3)

So far we collected unit step functions to express function from the graph.

$$L\{f(t)\}=L\{2u(t)-3u(t-2)+u(t-3)\}=L\{2u(t)\}-3L\{u(t-2)\}+L\{u(t-3)\}=\frac{2}{s}-\frac{3}{s}e^{-2s}+\frac{1}{s}e^{-3s}$$

Example: Determine the Laplace transform of the function

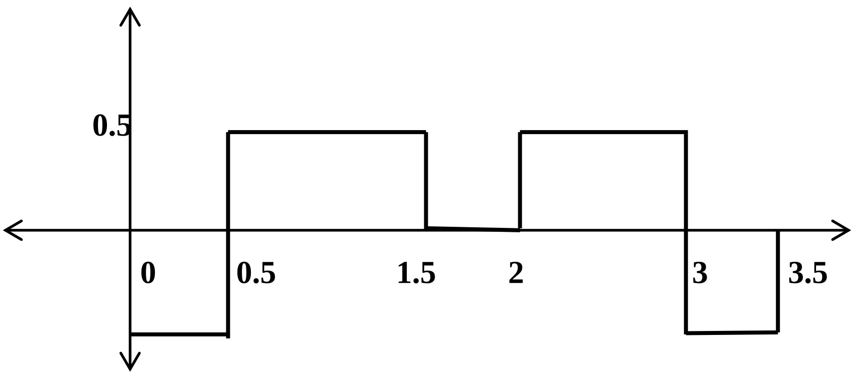
$$f(t)=\begin{cases}0 & t<1\\ 1 & 1\leq t\leq 3\\ 3 & 3\leq t\leq 5\\ 2 & 5\leq t\leq 6\\ 0 & t\geq 6\end{cases}$$



Solution:

$$f(t)=0u(t-0)+1u(t-1)+2u(t-3)-1u(t-5)-2u(t-6)$$
$$F(s)=\frac{1}{s}e^{-s}+\frac{2}{s}e^{-3s}-\frac{1}{s}e^{-5s}-\frac{2}{s}e^{-6s}=\frac{1}{s}(e^{-s}+2e^{-3s}-e^{-5s}-2e^{-6s})$$

Example:



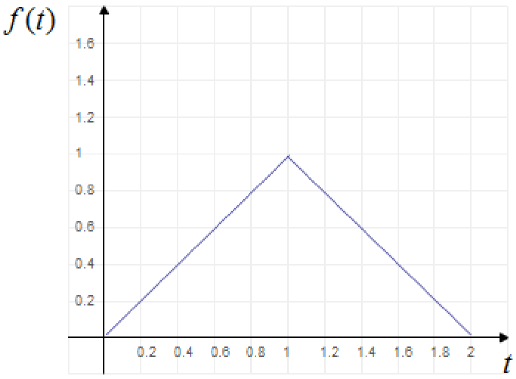
Solution:

$$f(t) = -0.5 + 1u(t-0.5) - 0.5u(t-1.5) + 0.5u(t-2) - 1u(t-3) + 0.5u(t-3.5)$$

$$f(s) = -\frac{0.5}{s} + \frac{1}{s}e^{-0.5s} - \frac{0.5}{s}e^{-1.5s} + \frac{0.5}{s}e^{-2s} - \frac{1}{s}e^{-3s} + \frac{0.5}{s}e^{-3.5s}$$

Example: Find F(s) for

$$f(t) = \begin{cases} 0 & t<0 \\ t & 0<t<1 \\ 2-t & 1<t<2 \\ 0 & t>2 \end{cases}$$



Solution:

$$f(t) = tU(t) - t(t-1)U(t-1) - (t-1)U(t-1) + (t-2)U(t-2)$$

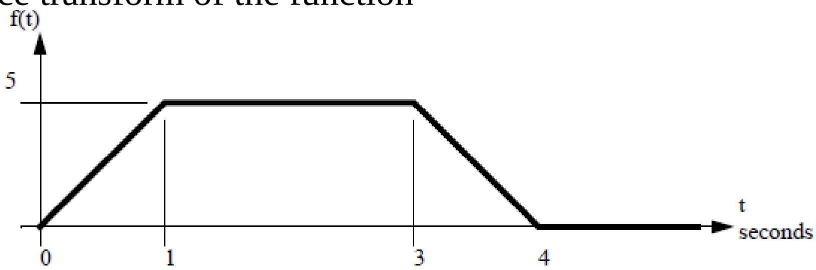
$$= tU(t) - 2(t-1)U(t-1) + (t-2)U(t-2)$$

$$L[f(t)] = L[tU(t) - 2(t-1)U(t-1) + (t-2)U(t-2)]$$

$$= L[tU(t)] - L[2(t-1)U(t-1)] + L[(t-2)U(t-2)]$$

$$= \frac{1}{s^2} - \frac{2}{s^2}e^{-s} + \frac{1}{s^2}e^{-2s}$$

Example: Determine the Laplace transform of the function



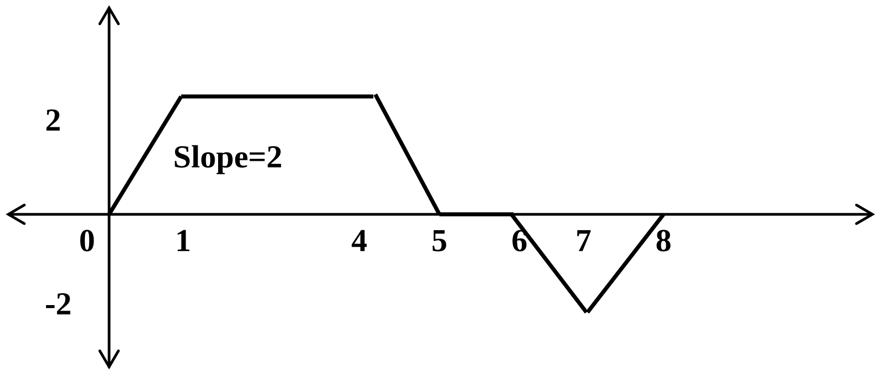
Solution:

$$f(t) = 5tu(t) - 5(t-1)u(t-1) - 5(t-3)u(t-3) + 5(t-4)u(t-4)$$

$$F(s) = \frac{5}{s^2} - \frac{5}{s^2}e^{-s} - \frac{5}{s^2}e^{-3s} + \frac{5}{s^2}e^{-4s} = \frac{5}{s^2}(1 - e^{-s} - e^{-3s} + e^{-4s})$$

<u>Example</u>: Find the Laplace transform of f(t) shown in Fig.

$$f(t)=\begin{cases}2t & 0\le t<1\\ 2 & 1\le t<4\\ -2t & 4\le t<5\\ 0 & 5\le t<6\\ -2t & 6\le t<7\\ 2t & 7\le t<8\\ 0 & t>8\end{cases}$$

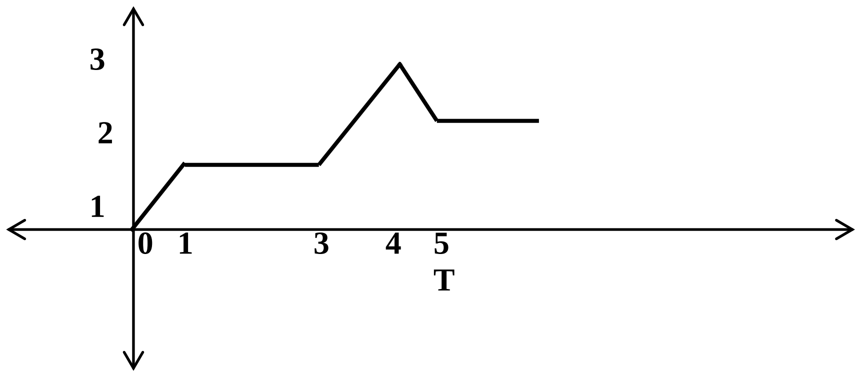


Solution:

$$f(t)=2tu(t)-2(t-1)u(t-1)-2(t-4)u(t-4)+2(t-5)u(t-5)-2(t-6)u(t-6)$$
$$+2\times 2(t-7)u(t-7)-2(t-8)u(t-8)$$

$$f(s)=\frac{2}{s^2}-\frac{2}{s^2}e^{-s}-\frac{2}{s^2}e^{-4s}+\frac{2}{s^2}e^{-5s}-\frac{2}{s^2}e^{-6s}+\frac{4}{s^2}e^{-7s}-\frac{2}{s^2}e^{-8s}$$

<u>Example:</u>



Solution:

$$f(t)=tu(t)-(t-1)u(t-1)+2(t-3)u(t-3)-2(t-4)u(t-4)+(t-5)u(t-5)$$

$$f(s)=\frac{1}{s^2}-\frac{1}{s^2}e^{-s}+\frac{2}{s^2}e^{-3s}-\frac{2}{s^2}e^{-4s}+\frac{1}{s^2}e^{-5s}$$

<u>2.6 Rational Functions Technique: Partial fraction</u>

Often it is necessary to break down a complicated rational function of the form $\frac{P(s)}{Q(s)}$ (where *P(s)* and *Q(s)* are polynomials in *s*, and the degree of the top polynomial is less than the degree of the bottom polynomial), into the sum of simpler fractions called ***Partial Fractions***.

The type of partial fraction that you use depends on the factors of the bottom polynomial**.**

We will look at 3 cases:
Case 1: All the factors of the bottom *Q(s)* are linear and non-repeating**.**
Case 2: *Q(s)* has some repeated linear factors.
Case 3: *Q(s)* has some irreducible quadratic factors.

Examples of case 1:

$\frac{s}{(s+2)(s-1)}$ or $\frac{3+2s}{s(s+5)(s+3)}$

Examples of case 2:

$\frac{2s+1}{(s-8)^2}$ or $\frac{2}{s(s-3)^2}$ or $\frac{s-1}{(s+1)^2(s+2)}$ or $\frac{s^2-1}{(s+3)^2(s-1)^3}$

Examples of case 3:

$\frac{2s+1}{s(s^2+4)}$ or $\frac{3}{((s-3)^2+1)(s+2)}$ or $\frac{4s^2+3s-1}{(s-1)(s^2+3)^2((s+1)^2+9)^2}$

<u>Case 1: All the factors of the bottom *Q(s)* are linear and non-repeating.</u>

Case 1: All the bottom factors are linear (i.e. of the form x $\pm$ some number) and then is no repeated factor (i.e. there is no factor which is squared or cubed, etc.) and there are no irreducible quadratic terms (don't worry about this!). In this case therefore, we are talking about rational functions of the form: $\frac{top\,polynomial}{(x-a)(x-b)...(x-g)}$

In this case we can rewrite the rational function as follows:

$$\boxed{\frac{top\,polynomial}{(x-a)(x-b)...(x-g)}=\frac{A}{x-a}+\frac{B}{x-b}+....+\frac{G}{x-g}}$$

Example

Show that $\frac{1}{(s+7)(s+3)}=\frac{-\frac{1}{4}}{s+7}+\frac{\frac{1}{4}}{s+3}$

Solution

This is a case 1 partial fraction so we start with

$$\frac{1}{(s+7)(s+3)}=\frac{A}{s+7}+\frac{B}{s+3}$$

<u>Step1: Remove Fractions</u>

Multiply both sides of the equation by the denominator on the left hand side

$$\frac{1\times(s+7)(s+3)}{(s+7)(s+3)}=\frac{A(s+7)(s+3)}{s+7}+\frac{B(s+7)(s+3)}{s+3}$$

$$\Rightarrow 1=A(s+3)+B(s+7)$$

<u>Step2: Choose s values to find A and B</u>

The equation above is true for ***all*** values of s. We can choose s values to make things simple:

Choose s = -3 so that (s+3) = 0 and we have

$$1=A(0)+B(4)\ \Rightarrow\ 1=4B\ \Rightarrow\ B=\frac{1}{4}$$

Choose s = -7 so that (s+7) = 0 and we have

$$1=A(-4)+B(0)\ \Rightarrow\ 1=-4A\ \Rightarrow\ A=-\frac{1}{4}$$

<u>Step3: Substitute A and B into the original expression</u>

$$\frac{1}{(s+7)(s+3)} = \frac{-\frac{1}{4}}{s+7} + \frac{\frac{1}{4}}{s+3}$$

Example: Write $\frac{3s-1}{(s+1)(s+2)}$ in partial fraction form

Solution:

write $\frac{3s-1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$

Multiply both sides by $(s+1)(s+2)$*:*

$$\frac{3s-1}{(s+1)(s+2)}(s+1)(s+2) = \frac{A}{s+1}(s+1)(s+2) + \frac{B}{s+2}(s+1)(s+2)$$

$$\Rightarrow 3s-1 = A(s+2) + B(s+1)$$

If choose s = -2

$$\Rightarrow 3(-2)-1 = A(-2+2) + B(-2+1) \Rightarrow -7 = A.0 + B(-1) \Rightarrow -7 = -B$$

$$\Rightarrow B = 7$$

If choose s = -1 $\Rightarrow 3(-1)-1 = A(-1+2) + B(-1+1) \Rightarrow -4 = A(1) + B(0) \Rightarrow -4 = A$

$$\Rightarrow A = -4$$

Therefore $\frac{3s-1}{(s+1)(s+2)} = \frac{-4}{s+1} + \frac{7}{s+2}$

Example

Find the fixed constants *A*, *B*, *C* so that the partial fraction decomposition can be completed: $\frac{4s+3}{s(s-1)(s+3)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+3}$

Solution:

$$\frac{4s+3}{s(s-1)(s+3)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+3}$$

Multiplying both sides of this equation by the left hand side denominator we deduce that,

$$s(s-1)(s+3)\frac{4s+3}{s(s-1)(s+3)} = s(s-1)(s+3)\frac{A}{s} + s(s-1)(s+3)\frac{B}{s-1} + s(s-1)(s+3)\frac{C}{s+3}$$

$$\Rightarrow 4s+3 = A(s-1)(s+3) + Bs(s+3) + Cs(s-1)$$

Now in turn, put , *s= 0* , *s=1, s = -3*. You will find at each stage that **2** of the *A, B, C* terms will vanish:

If choose *s* = 0

$$4(0)+3 = A(0-1)(0+3) + B(0) + C(0) \Rightarrow 3 = -3A \Rightarrow A = 1$$

If choose *s* = 1

$$4(1)+3 = A(0) + B(1)(1+3) + C(0) \Rightarrow 7 = 4B \Rightarrow B = \frac{7}{4}$$

If choose *s* = -3

$$4(-3)+3 = A(0) + B(0) + C(-3)(-3-1) \Rightarrow -9 = 12C \Rightarrow B = \frac{-9}{12}$$

Therefore :

$$\frac{4s+3}{s(s-1)(s+3)}=\frac{[-1]}{s}+\frac{\left[\frac{7}{4}\right]}{s-1}+\frac{\left[\frac{-3}{4}\right]}{s+3}=-\frac{1}{s}+\frac{7}{4(s-1)}-\frac{3}{4(s+3)}$$

Case 2: Denominator *Q(s)* has some repeated linear factors.

Sometimes a linear factor is repeated twice or three times or four times, etc. This means that we will have an expression like $(s-\text{some number})^2$ or $(s-\text{some number})^3$ or ... below the line. When this occurs, you have to be careful as we have to use a partial fraction for each of the powers. If you want, you can use the following table:

Factor in given rational function	Corresponding partial fraction
$\frac{top\ polynomial}{s\pm some\ number}$	$\frac{A}{s\pm some\ number}$
$\frac{top\ polynomial}{(s\pm somenumber)^2}$	$\frac{A}{s\pm some\ number}+\frac{B}{(s\pm some\ number)^2}$
$\frac{top\ polynomial}{(s\pm some\ number)^3}$	$\frac{A}{s\pm some\ number}+\frac{B}{(s\pm some\ number)^2}+\frac{C}{(s\pm some\ number)^3}$

Examples of case 2:

$$\frac{4s^2+3s-2}{(s-1)^2(s+3)}=\frac{A}{(s-1)^2}+\frac{B}{s-1}+\frac{C}{s+3}\ ,\quad A,B,C=\text{constants}$$

$$\frac{3s^4-7s^3+2s^2-5}{s^3(s+1)^2(s+5)}=\frac{A}{s^3}+\frac{B}{s^2}+\frac{C}{s}+\frac{D}{(s+1)^2}+\frac{E}{(s+1)}+\frac{F}{(s+5)}\quad A,B,C,D,E,F=\text{constants}$$

Example:

Write in terms of partial fractions $\frac{8s-1}{(s-1)^2}$

Solution:

In this case $\frac{8s-1}{(s-1)^2}=\frac{A}{s-1}+\frac{B}{(s-1)^2}$

Step1 we multiply both sides by $(s-1)^2$ to remove fractions

$$\frac{8s-1}{(s-1)^2}(s-1)^2=\frac{A}{s-1}(s-1)^2+\frac{B}{(s-1)^2}(s-1)^2\Rightarrow 8s-1=A(s-1)+B$$

Step2: we can choose s = 1 as before so that (s-1) = 0 and we get

$8-1=B\ \Rightarrow\ B=7$

We cannot choose another s value to directly find A however. There is more than one approach to finding A but the easiest method is called "***equating coefficients***". In this case, we note that there must be the same "amount of s" on both sides of the equation. On the left hand side we have 8s and on the right we have As, so that A must be 8.

Step3; write the answer down

$$\frac{8s-1}{(s-1)^2}=\frac{8}{s-1}+\frac{7}{(s-1)^2}$$

Example

Write $\frac{2s+1}{(s-1)(s+2)^2}$ in partial fraction form.

Solution:

As the linear factor (s+2) repeats (i.e. we have (s+2) and $(s+2)^2$ in the denominator) the required partial fraction is of the form,

$$\frac{2s+1}{(s-1)(s+2)^2}=\frac{A}{s-1}+\frac{B}{s+2}+\frac{C}{(s+2)^2}$$

where A, B and C are fixed constants which are to be found.

Step1: multiply both sides by the denominator $(s-1)(s+2)^2$

$$(s-1)(s+2)^2\frac{2s+1}{(s-1)(s+2)^2}=(s-1)(s+2)^2\frac{A}{s-1}+(s-1)(s+2)^2\frac{B}{s+2}+(s-1)(s+2)^2\frac{C}{(s+2)^2}$$

$$\Rightarrow 2s+1=(s+2)^2A+(s-1)(s+2)B+(s-1)C$$

The latter equation holds for **all** values of the variable s.

Step2: Choose good s values

Choose s = -2 $\Rightarrow 2(-2)+1=(0)^2A+(0)B+(-2-1)C \Rightarrow -3=-3C \;\Rightarrow\; C=1$

Choose s = 1 $\Rightarrow 2(1)+1=(1+2)^2A+(0)B+(0)C \Rightarrow 3=9A \;\Rightarrow\; A=\frac{1}{3}$

We again have the problem of not being able to choose a good s value to find B. We again "equate coefficients". The best strategy is to equate the highest power of s on both sides, which is s^2 . On the left hand side of (1) we have 0 lots of s^2. On the right hand side of (1) the $(s+2)^2A$ term will contribute As^2 if you multiply it out and the $(s-1)(s+2)B$ term will contribute Bs^2. So we have

$$0=A+B \qquad \Rightarrow B=-A=-\frac{1}{3}$$

This method can involve more calculation though. Either way

$$\frac{2s+1}{(s-1)(s+2)^2}=\frac{1}{3}.\frac{1}{s-1}-\frac{1}{3}.\frac{B}{s+2}+\frac{1}{(s+2)^2}$$

Case 3 Denominator *Q(s)* has some irreducible quadratic factors.

Irreducible means that the quadratic term in the denominator cannot be factorized into two brackets.

Examples of case 3:

$$\frac{3s^2-2s-5}{s(s^2+4)}=\frac{A}{s}+\frac{Bs+C}{s^2+4} \quad , \quad A,B,C=\text{constants}$$

$$\frac{s^3-s+2}{s(s-1)(s^2+2)}=\frac{A}{s}+\frac{B}{s-1}+\frac{Cs+D}{s^2+2} \quad , \quad A,B,C,D=\text{constants}$$

Example:

Write $\frac{5s^2-7s+8}{(s-1)(s^2-2s+5)}$ in partial fractions.

Solution

The denominator contains an irreducible quadratic term (i.e, it cannot be easily factored into two linear terms. If it did factor into linear factors then we would be back to cases 1 or 2.) As it does not in this example we must write:

$$\frac{5s^2-7s+8}{(s-1)(s^2-2s+5)}=\frac{A}{(s-1)}+\frac{Bs+C}{(s^2-2s+5)}$$

Note: Linear term on bottom means ***constant*** A on top. Quadratic term on bottom means ***linear*** term *Bs* + *C* on top.

As before to find these constants multiply both sides by the denominator (s – 1) (s^2-2s+5).

$$(s-1)(s^2-2s+5)\frac{5s^2-7s+8}{(s-1)(s^2-2s+5)}=(s-1)(s^2-2s+5)\frac{A}{(s-1)}+(s-1)(s^2-2s+5)\frac{Bs+C}{(s^2-2s+5)}$$

Which gives: $5s^2-7s+8=(s^2-2s+5)A+(s-1)(Bs+C)$

The latter equation holds for **all** values of the variable *s*. So we can choose any values for *s* and set up three simultaneous equations for *A*, *B* and *C*.

By putting *s* = 1 we can get one value easily:

Choose $s = 1$

$$\Rightarrow 5(1)^2-7(1)+8=(1^2-2(1)+5)A+0(B(1)+C)\Rightarrow 6=4A+0\Rightarrow A=\frac{3}{2}$$

We cannot make any more brackets = 0 by a good choice of s. As for case 2 however, we can equate coefficients. Start with the highest power s^2 first

Equate s^2 $\Rightarrow 5=A+B \Rightarrow B=5-A \Rightarrow B=5-\frac{3}{2}=\frac{7}{2}$

Equate s $\Rightarrow -7=-2A-B+C \Rightarrow -7=-2(\frac{3}{2})-\frac{7}{2}+C \Rightarrow -7=-\frac{13}{2}+C$

$\Rightarrow C=-\frac{1}{2}$

2.7 Inverse Laplace Transforms

The Laplace transform is an expression in the variable s which denoted by $F(s)$. It is said that $f(t)$ and $F(s)=L\{f(t)\}$ form a transform pair. This means that if $F(s)$ is the Laplace transform of $f(t)$ then $f(t)$ is the inverse Laplace transform of $F(s)$. We write as: $\boxed{f(t)=L^{-1}\{F(s)\} \quad \text{or} \quad L^{-1}\{F(s)\}=f(t)}$

The operator L^{-1} is known as the operatorfor inverse Laplace transform. There is no simple integral definition of the inverse transform so you have to find it by working backwords.

Here we have the reverse process, i.e. given a Laplace transform, we have to find the function of t to which it belongs. We use the following table:

Table of inverse transforms

$F(s)$	$f(t)$
$\frac{a}{s}$	a
$\frac{1}{s+a}$	e^{-at}

$\frac{n!}{s^{n+1}}$	t^n
$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$
$\frac{a}{s^2+a^2}$	$\sin at$
$\frac{s}{s^2+a^2}$	$\cos at$
$\frac{a}{s^2-a^2}$	$\sinh at$
$\frac{s}{s^2-a^2}$	$\cosh at$

2.7.1 Two Properties of Laplace Transform Inverse

Both Laplace transform and its inverse are linear transforms, by which is meant that:

i. The transform of a sum (or difference) of expressions is the sum (or difference of the individual transforms. That is:

$$L^{-1}\{F(s) \pm G(t)\} = L^{-1}\{F(s)\} \pm L^{-1}\{G(s)\}$$

ii. The transform of an expression that is multiplied by a constant is the constant multiplied by the transform of the expression. That is:

$L^{-1}\{kF(s)\} = kL^{-1}\{F(s)\}$ where k is constant

Example: find $L^{-1}\left\{\frac{1}{s-2}\right\}$?

Solution: $L^{-1}\left\{\frac{1}{s-2}\right\} = L^{-1}\left\{\frac{1}{s+(-2)}\right\} = e^{2t}$.

Example: find $L^{-1}\left\{\frac{8}{s^2+64}\right\}$?

Solution: We can write the inverse transform as we know that $L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$.

Here we have $a = 8$ therefore: $L^{-1}\left\{\frac{8}{s^2+64}\right\} = L^{-1}\left\{\frac{8}{s^2+8^2}\right\} = \sin 8t.$

Example: find $L^{-1}\left\{\frac{12}{s^2-9}\right\}$?

Solution: $L^{-1}\left\{\frac{12}{s^2-9}\right\} = 4L^{-1}\left\{\frac{3}{s^2-9}\right\} = 4\sinh 3t$

Example: Find the inverse Laplace transform for $F(s) = -\frac{2}{3s-4}$.

Solution:

$$L^{-1}\left\{-\frac{2}{3s-4}\right\}=L^{-1}\left\{-\frac{2}{3}\left(\frac{1}{s-4/3}\right)\right\}=-\frac{2}{3}L^{-1}\left\{\frac{1}{s-4/3}\right\}=-\frac{2}{3}e^{4/3t}$$

Example: Determine $L^{-1}\left\{\frac{3s+1}{s^2-s-6}\right\}$.

Solution: This certainly did not appear in our list of standard transforms but if we write $\frac{3s+1}{s^2-s-6}$ as the sum of two simpler functions, i.e.

$\frac{3s+1}{s^2-s-6}=\frac{1}{s+2}+\frac{2}{s-3}$ it makes all the difference.

This is simply the method of writing the more complex algebraic fraction in terms of its partial fractions which we have previously seen.

We can now proceed to find

$$L^{-1}\left\{\frac{3s+1}{s^2-s-6}\right\}=L^{-1}\left\{\frac{1}{s+2}+\frac{2}{s-3}\right\}=e^{-2t}+2e^{3t}$$

Example: Determine $L^{-1}\left\{\frac{5s+1}{s^2-s-12}\right\}$.

Solution: Factorise the denominator: $\frac{5s+1}{s^2-s-12}=\frac{5s+1}{(s-4)(s+3)}$.

Remember from partial fractions we have the form:

$$\frac{5s+1}{(s-4)(s+3)}=\frac{A}{(s-4)}+\frac{B}{(s+3)} \qquad \times(s-4)(s+3)$$

$$5s+1=A(s+3)+B(s-4)$$

Choose s = -3 ; $5(-3)+1=0A+(-3-4)B \Rightarrow B=\frac{-14}{-7}=2$

Choose s = 4 ; $5(4)+1=(4+3)A+0B \Rightarrow A=\frac{21}{7}=3$

This gives us $\frac{5s+1}{(s-4)(s+3)}=\frac{3}{(s-4)}+\frac{2}{(s+3)}$

So we now have to find

$$L^{-1}\left\{\frac{5s+1}{(s-4)(s+3)}\right\}=L^{-1}\left\{\frac{3}{(s-4)}+\frac{2}{(s+3)}\right\}=3L^{-1}\left\{\frac{1}{(s-4)}\right\}+2L^{-1}\left\{\frac{1}{(s+3)}\right\}$$

$$=3e^{4t}+2e^{-3t}$$

Example: Determine $L^{-1}\left\{\frac{9s-8}{s^2-2s}\right\}$.

Solution: Simplify: $L^{-1}\left\{\frac{9s-8}{s^2-2s}\right\}=L^{-1}\left\{\frac{9s-8}{s(s-2)}\right\}$

Remember from partial fractions we have the form:

$$\frac{9s-8}{s(s-2)}=\frac{A}{s}+\frac{B}{(s-2)} \qquad \times s(s-2)$$

$$9s-8=A(s-2)+Bs$$

Let s=0 then $A = \frac{0-8}{0-2} = 4$

Let s=2 then $B = \frac{9\times 2-8}{2} = 5$

This gives us $\frac{9s-8}{s(s-2)} = \frac{4}{s} + \frac{5}{(s-2)}$

So we now have to find

$$L^{-1}\left\{\frac{9s-8}{s(s-2)}\right\} = L^{-1}\left\{\frac{4}{s} + \frac{5}{(s-2)}\right\} = L^{-1}\left\{\frac{4}{s}\right\} + 5L^{-1}\left\{\frac{1}{(s-2)}\right\} = 4 + 5e^{2t}$$

Example: Determine $L^{-1}\left\{\frac{13s+11}{(s-1)(s+3)}\right\}$.

Solution: Remember from partial fractions we have the form:

$\frac{13s+11}{(s-1)(s+3)} = \frac{A}{(s-1)} + \frac{B}{(s+3)}$ Multiple both sides by $(s-1)(s+3)$

$13s+11 = (s+3)A + (s-1)B$

Let s=1 then $A = \frac{13+11}{1+3} = \frac{24}{4} = 6$

Let s=-3 then $B = \frac{-3\times 13+11}{-3-1} = \frac{-28}{-4} = 7$

So we now have to find

$$L^{-1}\left\{\frac{13s+11}{(s-1)(s+3)}\right\} = L^{-1}\left\{\frac{6}{(s-1)} + \frac{7}{(s+3)}\right\} = 6L^{-1}\left\{\frac{1}{(s-1)}\right\} + 7L^{-1}\left\{\frac{1}{(s+3)}\right\} = 6e^{t} + 7e^{-3t}$$

Example: Find the inverse Laplace transform of $F(s) = \frac{s+4}{(s+2)(s+1)^2}$

Solution: Expand F(s) as $F(s) = \frac{s+4}{(s+2)(s+1)^2} = \frac{A}{s+2} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)}$

$(s+4) = (s+1)^2 A + (s+2)B + (s+1)(s+2)C$

Let s=-1 then $B = \frac{-1+4}{-1+2} = \frac{3}{1} = 3$

Let s=-2 then $A = \frac{-2+4}{(-2+1)^2} = \frac{2}{1} = 2$

equate s $\Rightarrow 1 = 2A + B + 3C \Rightarrow\ 1 = 2\times 2 + 3 + 3C \Rightarrow C = \frac{1-7}{3} = -2$

Check by cross-multiplying:

$$\frac{s+4}{(s+2)(s+1)^2} = \frac{2}{(s+2)} + \frac{3}{(s+1)^2} - \frac{2}{(s+1)}$$

$$\frac{s+4}{(s+1)(s+3)^2} = \frac{2}{s+2} + \frac{3}{(s+1)^2} + \frac{-2}{(s+1)}$$

$$s+4 = 2(s^2+2s+1) + 3(s+2) - 2(s^2+3s+2)$$

$s^2: \quad 0 = 2-2$

$s^1: \quad 1 = 4+3-6$

$s^0: \quad 4 = 2+6-4$

$$L^{-1}\left\{\frac{s+4}{(s+2)(s+1)^2}\right\}=L^{-1}\left\{\frac{2}{(s+2)}+\frac{3}{(s+1)^2}-\frac{2}{(s+1)}\right\}$$

$$=2L^{-1}\left\{\frac{1}{(s+2)}\right\}+3L^{-1}\left\{\frac{1}{(s+1)^2}\right\}-2L^{-1}\left\{\frac{1}{(s+1)}\right\}=2e^{-2t}+3te^{-t}-2e^{-t}$$

Example: Find the inverse Laplace transform of $F(s)=\dfrac{s+9}{s^2+6s+13}$

Solution:

$$f(t)=L^{-1}[F(s)]=L^{-1}\left[\frac{s+9}{s^2+6s+13}\right]=L^{-1}\left[\frac{s+9}{s^2+6s+3^2-3^2+13}\right]=L^{-1}\left[\frac{s+9}{(s+3)^2+2^2}\right]$$

$$=L^{-1}\left[\frac{(s+3)+6}{(s+3)^2+2^2}\right]=L^{-1}\left[\frac{s+3}{(s+3)^2+2^2}\right]+L^{-1}\left[\frac{6}{(s+3)^2+2^2}\right]=L^{-1}\left[\frac{s+3}{(s+3)^2+2^2}\right]+L^{-1}\left[3\cdot\frac{2}{(s+3)^2+2^2}\right]$$

$$=L^{-1}\left[\frac{s+3}{(s+3)^2+2^2}\right]+3L^{-1}\left[\frac{2}{(s+3)^2+2^2}\right]=e^{-3t}\cos(2t)+3e^{-3t}\sin(2t)\;=e^{-3t}[\cos(2t)+3\sin(2t)]$$

Example: Find the Laplace inverse of $\dfrac{1}{s(s-1)}$ using

a) partial fraction b)convolution

Solution:

a) Partial fraction

$L^{-1}\left[\dfrac{1}{s(s-1)}\right]=L^{-1}\left[\dfrac{A}{s-1}+\dfrac{B}{s}\right]$ Multiple both sides by $s(s-1)$

$1=As+B(s-1)$

Let s=1 then A=1

Let s=0 then B=-1

$$L^{-1}\left[\frac{1}{s(s-1)}\right]=L^{-1}\left[\frac{1}{s-1}-\frac{1}{s}\right]=L^{-1}\left[\frac{1}{s-1}\right]-L^{-1}\left[\frac{1}{s}\right]=e^t-1$$

b) Convolution:

$$L^{-1}\left[\frac{1}{s(s-1)}\right]=L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s-1}\right]=1*e^t=\int_0^t 1e^{t-u}du=\int_0^t e^{t-u}du$$

$$=e^t\int_0^t e^{-u}du=e^t\left[-e^{-u}\right]_0^t=e^t\left[-e^{-t}+e^0\right]=e^t\left[1-e^{-t}\right]=e^t-1$$

Example: Find the Laplace inverse of $\dfrac{1}{s^2(s+1)^2}$ using convolution theorem

Solution:

$$L^{-1}\left[\frac{1}{s^2(s+1)^2}\right]=L^{-1}\left[\frac{1}{s^2}\cdot\frac{1}{(s+1)^2}\right]=t*te^{-t}=\int_0^t u(t-u)e^{-(t-u)}du$$

$$=e^{-t}\int_0^t u(t-u)e^{u}du\;=e^{-t}\left[t\int_0^t ue^u du-\int_0^t u^2e^u du\right]=e^{-t}\left[t\int_0^t ue^u du-\left[\left[u^2e^u\right]_0^t-\int_0^t 2ue^u du\right]\right]$$

$$= e^{-t}\left[t\int_0^t ue^u du - t^2e^{-t} + 2\int_0^t ue^u du\right] = e^{-t}\left[(t+2)\int_0^t ue^u du - t^2e^t\right] = e^{-t}\left[(t+2)\left[\left[ue^u\right]_0^t - \int_0^t e^u du\right] - t^2e^t\right]$$

$$= e^{-t}\left[(t+2)\left[\left[ue^u\right]_0^t - \left[e^u\right]_0^t\right] - t^2e^t\right] = e^{-t}\left[(t+2)\left[te^t - e^t + 1\right] - t^2e^t\right] = e^{-t}\left[t^2e^t - te^t + t + 2te^t - 2e^t + 2 - t^2e^t\right]$$

$$= e^{-t}\left[te^t + t - 2e^t + 2\right] = t + te^{-t} - 2 + 2e^{-t} = te^{-t} + 2e^{-t} + t - 2$$

2.8 Laplace Transform of a Derivative

Before we apply Laplace transform to solve a differential equation, we need to know the Laplace transform of a derivative. Given some expression $f(t)$ with Laplace transform $L\{f(t)\} = F(s)$, the Laplace transform of the derivative $f'(t)$ is:

$$L\{f'(t)\} = \int_{t=0}^{\infty} e^{-st} f'(t)dt$$

This can be integrated by parts as follows:

$$L\{f'(t)\} = \int_{t=0}^{\infty} e^{-st} f'(t)dt \qquad \text{where } u = e^{-st} \qquad dv = f'(t)$$

$$du = -se^{-st} \qquad v = \int f'(t) = f(t)$$

$$L\{f'(t)\} = \left[e^{-st} f(t)\right]_{t=0}^{\infty} + s\int_{t=0}^{\infty} e^{-st} f(t)dt = \left(0 - f(0)\right) + sF(s)$$

Assuming $e^{-st} f(t) \to 0$ as $t \to \infty$

That is: $\boxed{L\{f'(t)\} = sF(s) - f(0)}$

Thus, the Laplace transform of the derivative of $f(t)$ is given in terms of the Laplace transform of $f(t)$ when $t = 0$. The next properties is very important for the above formula.

In general, to solve differential equation $af'(t) + bf(t) = g(t)$ given that $f(0) = k$ where *a*, *b*, and *k* are known constants and $g(t)$ is a known expression in *t* using Laplace transform are as follows:

i. Take the Laplace transform of both sides of the differential equation.
ii. Find the expression of $F(s) = L\{f(t)\}$ in the form of an algebraic fraction
iii. Separate *F(s)* into its partial fractions.
iv. Find the inverse Laplace transform $L\{f'(t)\}$ to find the solution *f(t)* to the differential equation.

Example: Solve $f'(t) - f(t) = 2 \quad \text{where } f(0) = 0$

Solution: Taking Laplace transforms of both sides of the equation gives:

$$sF(s) - f(0) - F(s) = \frac{2}{s}$$

$$F(s)(s-1) = \frac{2}{s}$$

$$F(s)=\frac{2}{s(s-1)}$$ solve using partial fraction

$$\frac{2}{s(s-1)}=\frac{A}{s}+\frac{B}{s-1}$$ solve for A and B where $A=-2$ and $B=2$

$$F(s)=-\frac{2}{s}+\frac{2}{s-1}$$

The inverse transformation gives the solution as

$$f(t)=-2+2e^{t}=-2\left(1-e^{t}\right)$$

Example: Solve $f'(t)-f(t)=e^{2t}$ where $f(0)=1$

Solution:

$$sF(s)-f(0)-F(s)=\frac{1}{s-2}$$

$$F(s)(s-1)-1=\frac{1}{s-2}$$

$$F(s)=\frac{1}{(s-2)(s-1)}+\frac{1}{s-1}=\frac{(s-2)+1}{(s-2)(s-1)}=\frac{(s-1)}{(s-2)(s-1)}=\frac{1}{s-2}$$

The inverse transform then gives the solution as

$$f(t)=e^{2t}$$

Example: Solve $3f'(t)-2f(t)=4e^{-t}+2$ where $f(0)=0$

Solution:

$$3\left[sF(s)-f(0)\right]-2F(s)=\frac{4}{s+1}+\frac{2}{s}$$

$$3sF(s)-3f(0)-2F(s)=\frac{6s+2}{s(s+1)}$$

$$F(s)(3s-2)=\frac{6s+2}{s(s+1)}$$

$$F(s)=\frac{6s+2}{s(s+1)(3s-2)}$$ solve using partial fractions for $A=-1$, $B=-\frac{4}{5}$, and $C=\frac{27}{5}$

$$F(s)=-\frac{1}{s}-\frac{4}{5}\left(\frac{1}{s+1}\right)+\frac{27}{15}\left(\frac{1}{3s-2}\right)=-\frac{1}{s}-\frac{4}{5}\left(\frac{1}{s+1}\right)+\frac{81}{15}\left(\frac{1}{s-\frac{2}{3}}\right)$$

The inverse transform then gives the solution as :

$$f(t)=-1-\frac{4}{5}e^{-t}+\frac{81}{15}e^{\frac{2}{3}t}$$

3.8.1 Laplace Transforms of Higher Derivatives

The Laplace transforms of derivatives higher than the first are readily derived. To find higher derivative, understand the following formula

First order	$L\{f'(t)\} = sF(s) - F(0)$
Second order	$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$
Third order	$L\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$
Forth order	$L\{f^{iv}(t)\} = s^4F(s) - s^3f(0) - s^2f'(0) - sf''(0) - f'''(0)$

Example:

Find the solution of $f''(t) + 3f'(t) + 2f(t) = 4t$ where $f(0) = f'(0) = 0$.

Solution:

i. Take the Laplace transform of both sides of the equation

$$L\{f''(t)\} + 3L\{f'(t)\} + 2L\{f(t)\} = 4L\{t\}$$

$$[s^2F(s) - sF(0) - f'(0)] + 3[sF(s) - F(0)] + 2F(s) = \frac{4}{s^2}$$

ii. Find the expression $F(s) = L\{f(t)\}$ in the form of algebraic function

Substituting the values for $f(0)$ and $f''(0)$ and then rearranging the above equation gives

$$(s^2 + 3s + 2)F(s) = \frac{4}{s^2}$$

$$F(s) = \frac{4}{s^2(s+1)(s+2)}$$

iii. Separate $F(s)$ into its partial fractions

$$\frac{4}{s^2(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2}$$

$$4 = As(s+1)(s+2) + B(s+1)(s+2) + Cs^2(s+2) + Ds^2(s+1)$$

Let

$s = 0 \Rightarrow B = 2$

$s = -1 \Rightarrow C = 4$

$s = -2 \Rightarrow D = -1$

$equalize\ s^3 \Rightarrow 0 = A + C + D \Rightarrow A = 0 - C - D = 0 - 4 + 1 = -3$

Thus, $F(s) = -\frac{3}{s} + \frac{2}{s^2} + \frac{4}{s+1} - \frac{1}{s+2}$

iv. The inverse Laplace transform of the above equation is the solution that is

$$f(t) = -3 + 2t + 4e^{-t} - e^{-2t}$$

Example: Use Laplace transforms to solve the following D.E. with initial conditions.

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = 1 \quad , \quad y(0) = 1 \quad , \quad y'(0) = 1$$

Taking the Laplace

$$s^2Y(s) - sy(0) - y'(0) + 4[sY(s) - y(0)] + 3Y(s) = \frac{1}{s}$$

$$Y(s)[s^2 + 4s + 3] = sy(0) + y'(0) + 4y(0) + \frac{1}{s}$$

$$= s + 5 + \frac{1}{s}$$

We want to solve for Y(s), so

$$Y(s)=\frac{s+5+\frac{1}{s}}{s^2+4s+3}=\frac{s+5+\frac{1}{s}}{(s+3)(s+1)}=\frac{s^2+5s+1}{s(s+3)(s+1)}$$

Notice that we factored the denominator into individual terms. Once again, we use partial fraction expansion to break this down into terms we can look up in the table:

$$Y(s)=\frac{s^2+5s+1}{s(s+1)(s+3)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+3}$$

Multiple both sides by $s(s+1)(s+3)$

$$s^2+5s+1=(s+1)(s+3)A+s(s+3)B+s(s+1)C$$

Let s=0 $\Rightarrow A=\frac{1}{1\times 3}=\frac{1}{3}$

Let s=-1 $\Rightarrow B=\frac{(-1)^2-5+1}{-1(-1+3)}=\frac{-3}{-2}=\frac{3}{2}$

Let s=-3 $\Rightarrow C=\frac{(-3)^2-15+1}{-3(-3+1)}=\frac{-5}{6}$

So now we have

$$Y(s)=\frac{1/3}{s}+\frac{3/2}{s+1}+\frac{-5/6}{s+3}$$

and going back to the time domain gives

$$y(t)=\frac{1}{3}+\frac{3}{2}e^{-t}-\frac{5}{6}e^{-3t}$$

Example: Solve $\frac{d^2y(t)}{dt^2}+4\frac{dy(t)}{dt}+4y(t)=e^{-3t}u(t) \quad y(0)=1, y'(0)=0$

Taking the Laplace $s^2Y(s)-sy(0)+4[sY(s)-y(0)]+4Y(s)=\frac{1}{s+3}$

Which gives $Y(s)\left[s^2+4s+4\right]=s+4+\frac{1}{s+3}$

$$Y(s)=\frac{s+4+\frac{1}{s+3}}{(s+2)^2}$$

$$Y(s)=\frac{(s+4)}{(s+2)^2}+\frac{1}{(s+2)^2(s+3)}$$

This gives me two different terms, but they're simpler.

$$Y(s)=Y_1(s)+Y_2(s)$$

$$Y_1(s)=\frac{s+4}{(s+2)^2},\quad Y_2(s)=\frac{1}{(s+2)^2(s+3)}$$

First term:

$$Y_1(s)=\frac{s+4}{(s+2)^2}=\frac{s+2}{(s+2)^2}+\frac{2}{(s+2)^2}$$
$$=\frac{1}{(s+2)}+\frac{2}{(s+2)^2}$$
$$y_1(t)=e^{-2t}+2te^{-2t}$$

Now look at the second term:

$$Y_2(s)=\frac{1}{(s+2)^2(s+3)}=\frac{C}{s+3}+\frac{D}{(s+2)^2}+\frac{E}{(s+2)}$$
$$C=\left.\frac{1}{(s+2)^2}\right|_{s=-3}=\frac{1}{1^2}=1$$
$$D=\left.\frac{1}{(s+3)}\right|_{s=-2}=1$$
$$E=\frac{d}{ds}\left[\frac{1}{(s+3)}\right]\Bigg|_{s=-2}=\left[\frac{-1}{(s+3)^2}\right]\Bigg|_{s=-2}=\frac{-1}{1}=-1$$

Check:

$$Y_2(s)=\frac{1}{s+3}+\frac{1}{(s+2)^2}+\frac{-1}{(s+2)}=$$
$$=\frac{1(s^2+4s+4)+(s+3)-(s^2+5s+6)}{(s+3)(s+2)^2}=\frac{1}{(s+3)(s+2)^2}$$

so

$$y_2(t)=e^{-3t}+te^{-2t}-e^{-2t}$$

And putting the two solutions together:

$$y(t)=y_1(t)+y_2(t)=[2te^{-2t}+e^{-2t}]+[e^{-3t}+te^{-2t}-e^{-2t}]=e^{-3t}+3te^{-2t}$$

2.9 Solving Systems of Linear Differential Equations

Example:

Solve $y_1'=-y_1+y_2$, $y_2'=-y_1-y_2$, $y_1(0)=1$ and $y_2(0)=0$.

Solution:

Taken laplace of both equations

$sy_1(s)-y_1(0)=-y_1(s)+y_2(s)$ and $sy_2(s)-y_2(0)=-y_1(s)-y_2(s)$

$\Leftrightarrow$ $(s+1)y_1(s)-y_2(s)=1$ and $(s+1)y_2(s)+y_1(s)=0$

Solving for $y_1(s)$ and $y_2(s)$ algebraically we get

$(s+1)y_1(s)+\frac{y_1(s)}{s+1}=1$ and $(s+1)y_2(s)+y_1(s)=0$

$\Leftrightarrow(s+1)^2y_1(s)+y_1(s)=(s+1)$ and $y_2(s)=\left(-\frac{1}{s+1}\right)y_1(s)$

$\Leftrightarrow$ $[(s+1)^2+1]y_1(s)=(s+1)$ and $y_2(s)=\left(-\frac{1}{s+1}\right)y_1(s)$

$\Leftrightarrow\ y_1(s)=\dfrac{s+1}{(s+1)^2+1}$ and $y_2(s)=-\dfrac{1}{(s+1)^2+1}$

Now, $L^{-1}\left[\dfrac{s+1}{(s+1)^2+1}\right] = e^{-t}\cos t$ and $L^{-1}\left[\dfrac{1}{(s+1)^2+1}\right] = e^{-t}\sin t$

Therefore, $y_1(t)=e^{-t}\cos t$ and $y_2(t)=-e^{-t}\sin t$.

Example: Solve $y_1'=5y_1+y_2$, $y_2'=y_1+5y_2$, $y_1(0)=-3$ and $y_2(0)=7$.

Solution:

Taken Laplace of both equations

$sy_1(s)-y_1(0)=5y_1(s)+y_2(s)$ and $sy_2(s)-y_2(0)=y_1(s)+5y_2(s)$

$\Leftrightarrow\quad (s-5)y_1(s)-y_2(s)=-3$ and $(s-5)y_2(s)-y_1(s)=7$

Solving for $y_1(s)$ and $y_2(s)$ algebraically we get

$\Leftrightarrow\quad (s-5)y_1(s)-\dfrac{7-y_1(s)}{(s-5)}=-3$ and $y_2(s)=\dfrac{7+y_1(s)}{(s-5)}$

$\Leftrightarrow\quad (s-5)^2y_1(s)-7-y_1(s)=-3(s-5)$ and $y_2(s)=\dfrac{7+y_1(s)}{(s-5)}$

$\Leftrightarrow\ \left[(s-5)^2-1\right]y_1(s)=7-3(s-5)$ and $y_2(s)=\dfrac{7+y_1(s)}{(s-5)}$

$\Leftrightarrow\ y_1(s)=\dfrac{7}{(s-5)^2-1}-3\left[\dfrac{s-5}{(s-5)^2-1}\right]$ and $y_2(s)=\dfrac{7}{s-5}+\dfrac{7}{(s-5)\left((s-5)^2-1\right)}-\left[\dfrac{3}{(s-5)^2-1}\right]$

$\dfrac{7}{(s-5)\left((s-5)^2-1\right)}=\dfrac{A}{s-5}+\dfrac{B+sC}{(s-5)^2-1}$ multiple both side $(s-5)\left((s-5)^2-1\right)$

$7=A\left((s-5)^2-1\right)+(B+sC)(s-5)$

Let s=5 $\quad\Rightarrow A=-7$

Let s=0 $\quad\Rightarrow 7=A(25-1)-5B\Rightarrow B=\dfrac{-168-7}{5}=-35$

$s^2\Rightarrow 0=A+C\Rightarrow C=-A=7$

$y_2(s)=\dfrac{7}{s-5}-\dfrac{7}{(s-5)}+\dfrac{7s-35}{\left((s-5)^2-1\right)}-\left[\dfrac{3}{(s-5)^2-1}\right]$

$y_2(s)=7\left[\dfrac{s-5}{(s-5)^2-1}\right]-\left[\dfrac{3}{(s-5)^2-1}\right]$.

Now, $L^{-1}\left[\dfrac{s-5}{(s-5)^2-1}\right]=e^{5t}\cosh t$ and $L^{-1}\left[\dfrac{1}{(s-5)^2-1}\right]=e^{5t}\sinh t$

Therefore, $y_1(t)=e^{5t}(7\sinh t-3\cosh t)$ and $y_2(t)=e^{5t}(7\cosh t-3\sinh t)$.

Example: Solve $y_1''=y_1+3y_2$, $y_2''=4y_1-4e^t$, $y_1(0)=2$, $y_1'(0)=3$, $y_2(0)=1$ and $y_2'(0)=2$

Solution:

The subsidiary equations become

$s^2 y_1(s) - s\, y_1(0) - y_1'(0) = y_1(s) + 3y_2(s)$ and $s^2 y_2(s) - s\, y_2(0) - y_2'(0) = 4y_1(s) - \frac{4}{s-1}$

$\Leftrightarrow \quad (s^2-1)y_1(s) - 3y_2(s) = 3+2s$ and $s^2 y_2(s) - 4\,y_1(s) = 2+s-\frac{4}{s-1}$

Solving for $y_1(s)$ and $y_2(s)$ algebraically we get

$$y_1(s) = \frac{3+2s+3y_2(s)}{(s^2-1)} \text{ and } s^2 y_2(s) - \frac{12+8s+12y_2(s)}{(s^2-1)} = 2+s-\frac{4}{s-1}$$

$$y_1(s) = \frac{3+2s+3y_2(s)}{(s^2-1)} \text{ and } s^2(s^2-1)y_2(s) - 12 - 8s - 12y_2(s) = (s^2-1)(2+s) - \frac{4(s^2-1)}{s-1}$$

$$y_1(s) = \frac{3+2s+3y_2(s)}{(s^2-1)} \text{ and } [s^2(s^2-1)-12]y_2(s) = +12+8s+2s^2+s^3-2-s-4(s+1)$$

$$y_1(s) = \frac{3+2s+3y_2(s)}{(s^2-1)} \text{ and } [s^4-s^2-12]y_2(s) = s^3+2s^2+3s+6$$

$$y_1(s) = \frac{3+2s+3y_2(s)}{(s^2-1)} \text{ and } y_2(s) = \frac{s^3+2s^2+3s+6}{[s^4-s^2-12]} = \frac{1}{(s-2)}$$

$$y_1(s) = \frac{3+2s+3\frac{1}{(s-2)}}{(s^2-1)} = \frac{(3+2s)(s-2)+3}{(s-2)(s^2-1)} = \frac{3s-6+2s^2-4s+3}{(s-2)(s^2-1)} = \frac{2s^2-s-3}{(s-2)(s^2-1)}$$

$$= \frac{s^2+s^2-s-1-2}{(s-2)(s^2-1)} = \frac{(s^2-s-2)+(s^2-1)}{(s-2)(s^2-1)} = \frac{(s+1)(s-2)+(s^2-1)}{(s-2)(s^2-1)} = \frac{(s+1)}{(s^2-1)} + \frac{1}{(s-2)}$$

$$= \frac{1}{(s-1)} + \frac{1}{(s-2)}$$

$$\Leftrightarrow y_2(s) = \frac{1}{s-2} \text{ and } y_1(s) = \frac{1}{s-1} + \frac{1}{s-2}$$

Therefore, $y_1(t) = e^t + e^{2t}$ and $y_2(t) = e^{2t}$.

2.10 Laplace Transform of a integers

If f(t) is a function having Laplace transform F(s)= L{f(t)}, then the Laplace transform of the integration of a function f(t), is given by:

$$L\int_0^t f(t)dt = \frac{F(s)}{s}$$

Proof:

$$L\int_0^t f(t)dt = \int_0^\infty \left(\int_0^t f(t)\right) e^{-st}dt = \left[\frac{-1}{s}e^{-st}\int_0^t f(t)\right]_0^\infty + \frac{1}{s}\int_0^t f(t)e^{-st}dt = \frac{1}{s}F(s)$$

2.11 Initial value theorem:

It can be used to find the steady-state value of a system (providing that a steady-state value exists).

if $LF(t) = F(s)$, then

$$F(0) = \lim_{t \to 0} F(t) = \lim_{s \to \infty} sF(s)$$

2.12 Final value theorem

$$\lim_{t \to \infty} F(t) = \lim_{s \to 0} sF(s)$$

Example:

Find the final value of the function x(t) for which the laplace inverse is:-

$$x(s) = \frac{1}{s(s^3 + 3s^2 + 3s + 1)}$$

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sx(s) = \lim_{s \to 0} \frac{s \times 1}{s(s^3 + 3s^2 + 3s + 1)}$$

$$= \lim_{s \to 0} \frac{1}{(s^3 + 3s^2 + 3s + 1)} = 1$$

Example:

Suppose $Y(s) = \dfrac{5s + 2}{s(5s + 4)}$

Then steady state value of Y can be calculated by:

$$Y(\infty) = \lim_{t \to \infty} Y(t) = \lim_{s \to 0} \left[s \frac{(5s + 2)}{s(5s + 4)} \right] = 0.5$$

Response of first order systems

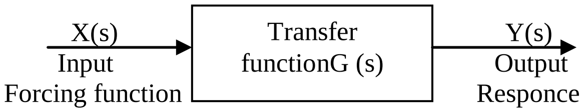
3. Dynamic behavior of first order system

Before studying the control system it is necessary to become familior with the response os some of simple basic systems (i.e study the dynamic behaviour of the first and second order systems).

3.1 The transfer function:

The dynamic behaviour of the system is described by transfer function (T.F)

$$\mathrm{T.F} = \frac{\text{Laplace transform of the output (responce)}}{\text{Laplace transform of the input (forcing function distubance)}}$$



Block diagram

$$\mathrm{T.F} = \mathrm{G(s)} = \frac{\mathrm{y(s)}}{\mathrm{x(s)}}$$

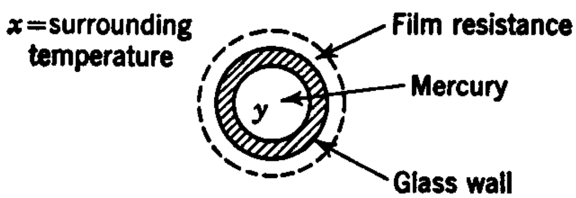
This definition is applied to linear systems

3.2 Development of T.F for first order system:

Mercury Thermometer:

It is a measuring device use to measure the temperature of a stream.

Consider a mercury in glass thermometer to be located in a flowing stream of fluid for which the temperature x varies with time.



The opject is to calculate the time variation of the thermometer reading y for a particular change of x

The following assumptions will be used in this analysis:-

1. All the resistance to heat transfer resides in the film surrounding the bulb (i.e., the resistance offered by the glass and mercury is neglected).

2. All the thermal capacity is in the mercury. Furthermore, at any instant the mercury assumes a uniform temperature throughout.

3. The glass wall containing the mercury does not expand or contract during the transient response.

It is assumed that the thermometer is initially at steady state. This means that, before time zero, there is no change in temperature with time. At time zero the thermometer will be subjected to some change in the surrounding temperature x(t). (i.e at t<0 , x(t)= y(t) =constant there is no change in temperature with time).

At t=0 there is a change in the surounding temperature x(t)

Unsteady state energy balance:

$$m\ Cp\ \frac{dy}{dt} = h\ A\ (x - y) - 0 = h\ A\ x - h\ A\ y \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

1st order differential equation

Where

A: area of the bulb

Cp: heat capacity of mercury

m: mass of mercury in the bulb

t: time

h: film heat transfer coefficient

h depend on the flowrate and properties of the surounding fluid and the dimension of the bulb.

The dynamic behaviour must be defined by a deviation variables.

At steady state (s.s.) , t<0 , x(t)=constant=x_s , y(t)=constant=y_s , x(t)=constant=x_s

$$m\ cp\ \frac{dy_s}{dt} = h\ A\ (x_s - y_s s) = h\ Ax_s - h\ Ay_s \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Substract eq. (2) from eq. (1)

$$m\ cp\ \frac{d(y - y_s)}{dt} = h\ A\ (x - x_s) - h\ A\ (y - y_s s)$$

$$y - y_s = Y \quad also \quad x - x_s = X$$

at t=0 Y(0)=0 and X(0)=0

$$m\ cp\ \frac{dY}{dt} = h\ A\ X - h\ A\ Y$$

$$\frac{m\ cp}{h\ A}\frac{dY}{dt} = X - Y$$

Let $\tau = \dfrac{m\ cp}{h\ A}$=time constant and has time units

$$\tau\ \frac{dY}{dt} + Y = X \quad \text{taken laplace for the equation}$$

$$\tau[sY(s) - Y(0)] + Y(s) = X(s)$$

$$(\tau s + 1)Y(s) = X(s)$$

$$\frac{Y(s)}{X(s)} = G(s) = \frac{1}{\tau s + 1} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\text{T.F} = \frac{\text{Y(s)}}{\text{X(s)}} = \text{G(s)} = \frac{\text{L.T of the deviation in thermometer reading}}{\text{L.T. of the deviation in surounding Temperature}}$$

Any system has a T.F of the form of equation (3) it is called *first order* system which is a first order differntial equation (Linear).

3.3 Properties of transfer functions

T.F relates two variables in a physical process. One of these is (Forcing or Input variable) and the other is the effect (Reponce or Output).

$$T.F = \frac{Y(s)}{X(s)} = G(s)$$

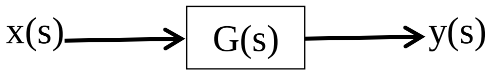
If we select a particular input variation x(t) for which the L.T is x(s) then the reponce.

$$Y(s) = G(s).X(s)$$

$$L^{-1}Y(s) = Y(t) = L^{-1}G(s).X(s)$$

If G(s) is 1st order of a thermometer

$$Y(s) = G(s).X(s) = \frac{1}{\tau s + 1}.X(s)$$

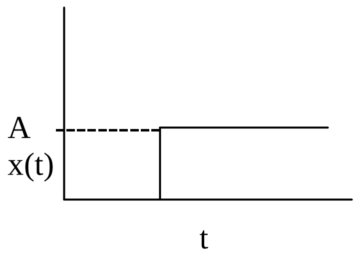


3.4 Transient response for different changes

$$Y(s) = \frac{1}{\tau s + 1}.X(s)$$

Y(t)=? For different types of x(t)

1-Step Change



$$X(s) = \frac{A}{s}$$

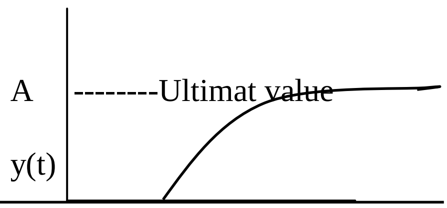
$$Y(s) = \frac{1}{\tau s + 1}.\frac{A}{s} = \frac{\alpha_o}{s} + \frac{\alpha_1}{\tau s + 1}$$

$$A = \alpha_o(\tau s + 1) + \alpha_1 s$$

$$S = 0 \quad \Rightarrow \alpha_o = A$$

$$s = -1/\tau \ \Rightarrow A = \alpha_o(-\tau/\tau + 1) - \alpha_1 \frac{1}{\tau} \text{ then } \alpha_1 = -A\tau$$

$$Y(s) = \frac{A}{s} - \frac{A\tau}{\tau s + 1} = \frac{A}{s} - \frac{A\tau}{\tau s + 1}\frac{1/\tau}{1/\tau} = \frac{A}{s} - \frac{A}{\tau + 1/\tau}$$



$Y(t) = A - Ae^{-t/\tau} = A(1 - e^{-t/\tau})$

Several features of this response, worth remembering, are

- The value of y(t) reaches 63.2 % of its ultimate value when the time elapsed is equal to one time constant τ.
- When the time elapsed is 2τ, 3 τ, and 4 τ, the percent response is 86.5%, 95%, and 98%, respectively.

Where the ultimate value is final steady state value

$$U.V = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sy(s)$$

Example:

A thermometer having a time constant of 0.1 min is at a steady state temperature of 90 F°. At time t = 0, the thermometer is placed in a temperature bath maintained at 100°F. Determine the time needed for the thermometer to read 98 F°.

Solution:

At s.s. $x_s = y_s = 90$ F°

$$Step\ change \quad X(s) = \frac{A}{s}$$

A=100-90=10

$$X(s) = \frac{10}{s}$$

$$Y(s) = \frac{1}{\tau s + 1}\frac{A}{s} = \frac{1}{0.1s + 1}\frac{10}{s} = \frac{10}{s(0.1s + 1)} = \frac{10}{0.1s(s + 10)} = \frac{A}{0.1s} + \frac{B}{s + 10}$$

$$A(s + 10) + B(0.1s) = 10$$

$$s = 0 \Rightarrow A = \frac{10}{10} = 1$$

$$s = -10 \Rightarrow B = -10$$

$$Y(s) = \frac{1}{0.1s} - \frac{10}{s + 10} = \frac{10}{s} - \frac{10}{s + 10}$$

By taken laplace inverse for the equation

$$Y(t) = 10 - 10e^{-10t} = 10(1 - e^{-10t})$$

Substitute Y(t)=y(t)-y_s =98-90

Y(t)=8

$$8 = 10(1 - e^{-10t})$$

$$0.8 = 1 - e^{-10t}$$

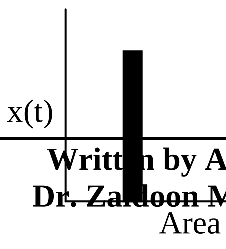
$$ln(e^{-10t}) = ln(0.2)$$

$$-10t = ln(0.2)$$

$$t = -\ln(0.2) \times 0.1$$

t=0.161 min

2-Impulse Input



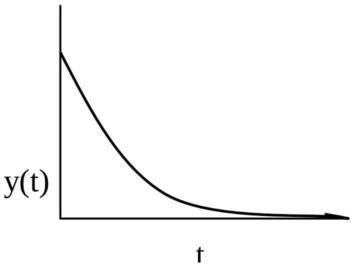
$X(s) = A$

$Y(s) = \frac{1}{\tau s + 1} \times A = \frac{A}{\tau s + 1}$

$Y(s) = \frac{A/\tau}{s + 1/\tau}$

$Y(t) = y(t) - y_s = \frac{A}{\tau} e^{-t/\tau}$

$y(t) = \frac{A}{\tau} e^{-t/\tau} + y_s$



3-Sinsoidal input

$x(t) = x_s + A\sin wt \qquad t > 0$

$x(t) - x_s = A\sin wt$

$X(t) = x(t) - x_s = A\sin wt$

$X(s) = \frac{Aw}{s^2 + w^2}$

$Y(s) = \frac{Aw}{s^2 + w^2} \times \frac{1}{(\tau s + 1)} = Aw[\frac{1}{(s^2 + w^2)(\tau s + 1)}]$

This equation can be solved for y(t) by means of a partial fraction expansion as described in previous lectures.

$Y(s) = Aw[\frac{1}{(s^2 + w^2)(\tau s + 1)}] = Aw[\frac{\alpha_o s + \alpha_1}{(s^2 + w^2)} + \frac{\alpha_2}{(\tau s + 1)}]$

$(\alpha_o s + \alpha_1)(\tau s + 1) + \alpha_2(s^2 + w^2) = 1$

$\alpha_o \tau s^2 + \alpha_o s + \alpha_1 \tau s + \alpha_1 + \alpha_2 s^2 + \alpha_2 w^2 = 1$

$s^0 \qquad \alpha_1 + \alpha_2 w^2 = 1 \qquad (4)$

$s^1 \qquad \alpha_o + \alpha_1 \tau = 0 \Rightarrow \alpha_o = -\alpha_1 \tau \qquad (5)$

$s^2 \qquad \alpha_o \tau + \alpha_2 = 0 \Rightarrow \alpha_2 = -\alpha_o \tau \qquad (6)$

By substitution eq.(5) in eq.(6)

$\alpha_2 = \alpha_1 \tau^2 \qquad (7)$

By substitution eq.(7) in eq.(4)

$\alpha_1 + \alpha_1 \tau^2 w^2 = 1$

$\alpha_1 = \frac{1}{1 + \tau^2 w^2}$

$$\alpha_2 = \frac{\tau^2}{1+\tau^2 w^2}$$

$$\alpha_0 = \frac{-\tau}{1+\tau^2 w^2}$$

$$Y(s) = Aw[\frac{\frac{-\tau}{1+\tau^2 w^2}s + \frac{1}{1+\tau^2 w^2}}{(s^2+w^2)} + \frac{\frac{\tau^2}{1+\tau^2 w^2}}{(\tau s+1)}]$$

$$Y(s) = Aw\frac{1}{1+\tau^2 w^2}[\frac{-\tau s+1}{(s^2+w^2)} + \frac{\tau^2}{(\tau s+1)}]$$

$$Y(s) = \frac{Aw}{1+\tau^2 w^2}[\frac{-\tau s}{(s^2+w^2)} + \frac{1}{(s^2+w^2)}\frac{w}{w} + \frac{\tau^2}{(\tau s+1)}\frac{\tau}{\tau}]$$

$$Y(s) = \frac{Aw}{1+\tau^2 w^2}[\frac{-\tau s}{(s^2+w^2)} + \frac{1}{w}\frac{w}{(s^2+w^2)} + \frac{\tau}{(s+1/\tau)}]$$

$$Y(t) = \frac{Aw}{1+\tau^2 w^2}[-\tau\cos wt + \frac{1}{w}\sin wt + \tau e^{t/\tau}]$$

Using the definition

$$p\cos\theta + q\sin\theta = r\sin(\theta+\phi)$$

$$r = \sqrt{p^2+q^2} \qquad \tan\phi = \frac{p}{q}$$

$$\phi = \tan^{-1}\frac{p}{q}$$

$$q = \frac{1}{w} \qquad p = -\tau$$

$$\phi = \tan^{-1}(-w\tau)$$

$$r = \sqrt{(\frac{1}{w})^2 + (-\tau)^2} = \sqrt{\frac{1}{w^2}+\tau^2} = \frac{\sqrt{1+w^2\tau^2}}{w}$$

$$-\tau\cos wt + \frac{1}{w}\sin wt = r\sin(wt+\phi)$$

$$Y(t) = \frac{Aw}{1+\tau^2 w^2}[\tau e^{t/\tau} + \frac{\sqrt{1+w^2\tau^2}}{w}\sin(wt+\phi)]$$

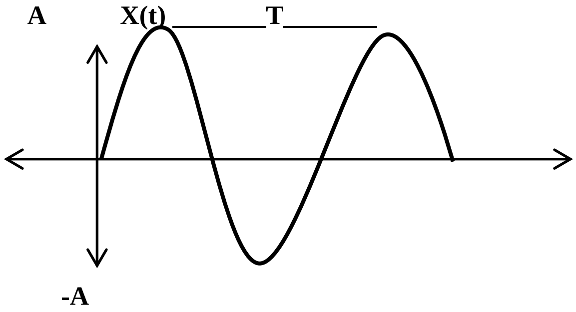
$$Y(t) = \frac{Aw\tau}{1+\tau^2 w^2}e^{t/\tau} + \frac{A}{\sqrt{1+w^2\tau^2}}\sin(wt+\phi)]$$

where

$$\phi = \tan^{-1}(-w\tau)$$

As $t \to \infty$ *then* $e^{-t/\tau} = 0$, the first term on the right side of main equation vanishes and leaves only the ultimate periodic solution, which is sometimes called the steady-state solution

$$Y(t) = \frac{A}{\sqrt{1+w^2\tau^2}} sin(wt + \phi)]$$



By comparing Eq. for the input forcing Y(t) function with Eq. for the ultimate periodic response X(t), we see that

1. The output is a sine wave with a frequency w equal to that of the input signal.
2. The ratio of output amplitude to input amplitude is $\frac{1}{\sqrt{1+w^2\tau^2}} < 1$.
3. The output lags behind the input by an angle ϕ. It is clear that lag occurs, for the sign of ϕ is always negative.

$\phi < 0$ *phase lag*

$\phi > 0$ *phase load*

Example:

A mercury thermometer having a time constant of 0.1 min is placed in a temperature bath at 100°F and allowed to come to equilibrium with the bath. At time t= 0, the temperature of the bath begins to vary sinusoidally about its average temperature of l00°F with an amplitude of 2°F If the frequency of oscillation is $10/\pi$ cycles/min, plot the ultimate response of the thermometer reading as a function of time. What is the phase lag?

In terms of the symbols used in this chapter

$\tau = 0.1$

$t < 0$ $x_s = y_s = 100$

$t \geq 0$ $x(t) = 100 + 2 sin(wt)$

$f = \frac{10}{\pi}$

Solution

$$w = 2\pi f = 2\pi \times \frac{10}{\pi} = 10 \ rad/\min$$

$$T = \frac{1}{f} = \frac{10}{\pi} \ \min/cycle$$

$$X(t) = x(t) - x_s = 100 + 2\sin 20t - 100$$

$$X(t) = 2\sin 20t$$

$$X(s) = \frac{2 \times 20}{s^2 + 20^2}$$

Ultimate response $t \to \infty$ *then* $e^{-t/\tau} = 0$

$$Y(t) = \frac{A}{\sqrt{1 + w^2\tau^2}} sin(wt + \phi)]$$

$$\phi = tan^{-1}(-w\tau) = tan^{-1}(-20 \times 0.1) = tan^{-1}(-2)$$

$$\phi = -63.5^o$$

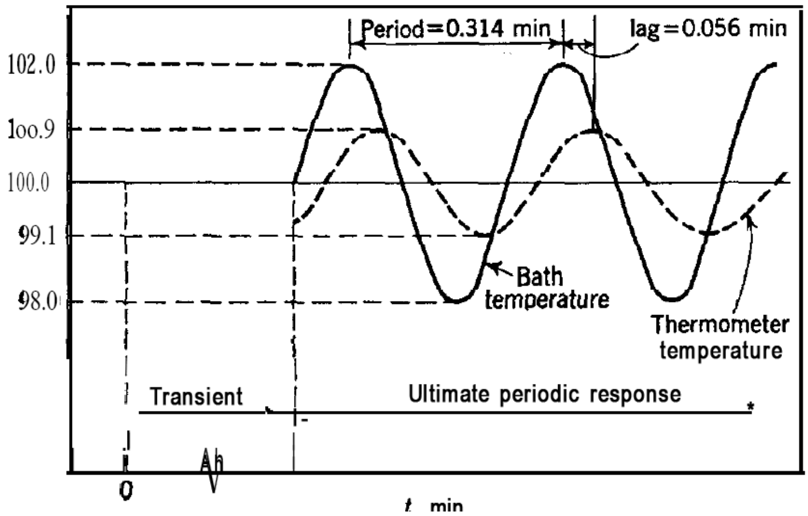
Ultimate response at the above angle

$$Y(t) = \frac{2}{\sqrt{1 + (0.1 \times 20)^2}} sin(20t - 63.5)]$$

$$Y(t) = \frac{2}{\sqrt{5}} sin(20t - 63.5)]$$

$$Y(t) = 0.896 \, sin(20t - 63.5)]$$

Ultimate response



In general, the lag in units of time is given by:-

$$phase \ lag = \frac{|\phi|}{360} \frac{1}{f}$$

$$phase \ lag = \frac{63.5 \ cycle}{360} \frac{\pi}{10} \frac{min}{cycle}$$

$$= 0.0555 \, min$$

A frequency of $\frac{10}{\pi}\frac{cycle}{min}$ means that a complete cycle occurs in $(\frac{10}{\pi})^{-1}$ min. since cycle is equivalent to 360^o and lag is 63.5^o

How to calculate the time constant (τ) for first order system

1) **Mathematical method**

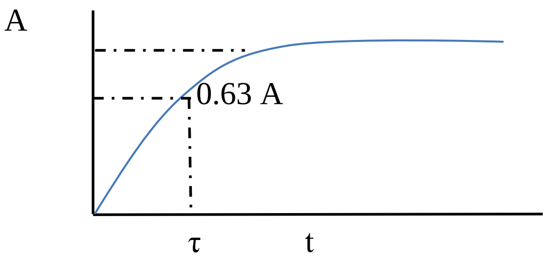
   Using the definitions

$$\tau = \frac{m\ cp}{hA} \qquad Thermometer$$

$$\tau = AR \qquad Liquid\ level\ \tan k$$

$$\tau = \frac{V}{q} \qquad Mixing\ \tan k$$

2) Exponential method(Step change in the input variable)



$$Y(t) = A(1 - e^{-t/\tau})$$

$$as\ t \to \infty \qquad Y(\infty) = A(1 - e^{-\infty}) = A$$

$$as\ t \to \tau \qquad Y(\tau) = A(1 - e^{-1}) = A(1 - 0.3678) \approx 0.63A$$

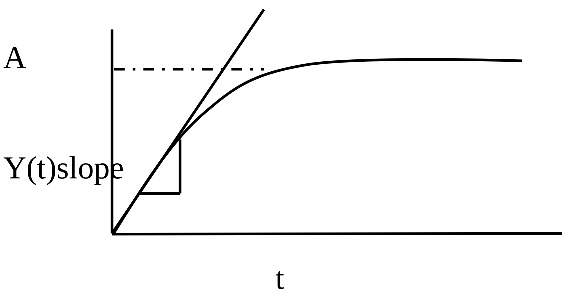
Time constant $(\tau)$is the time required for the response to reach 63% of the its utimate value.

3. Third method

$$Y(t) = A(1 - e^{-t/\tau})$$

$$\frac{dy}{dt} = -Ae^{-t/\tau}(-1/\tau) = \frac{A}{\tau}e^{-t/\tau}$$

$$\lim_{t\to 0}\frac{dy}{dt} = \frac{A}{\tau}e^{0} = \frac{A}{\tau}$$



Slope of the tangent at t=0 is $\frac{A}{\tau}$

Therfore $\tau = \frac{A}{slope}$

$$Y(t) = A(1 - e^{-t/\tau})$$

$$Y(t) = A - Ae^{-t/\tau}$$

$$Ae^{-t/\tau} = A - Y(t)$$

$$e^{-t/\tau} = \frac{A - Y(t)}{A}$$

$$-t/\tau = ln\frac{A - Y(t)}{A}$$

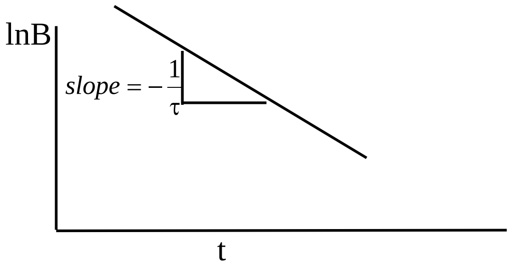
Let $B = \frac{A - Y(t)}{A}$

$$\ln B = \frac{-1}{\tau} t$$

$$let\ y = \ln B \quad and\ x = t$$

$$slope = -\frac{1}{\tau}$$

Y(t)	$B = \dfrac{A - Y(t)}{A}$	$\ln B$	t



$$slope = -\frac{1}{\tau}$$

$$\tau = -\frac{1}{slope}$$

Physical Examples of First Order System

1.Mercury Thermometer

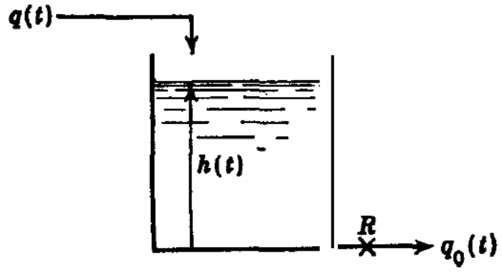
Mercury thermometer was discussed in detail in a previous lecture.

2.Liquid Level Tank

Consider the system shown in below figure which consists of:

1. A tank of uniform cross-sectional area *A*.
2. Valve attached to the output flow which resistance constant=R.

q_o: The output volumetric flowrate (volume/time) through the resistance, is related to the head *h* by the linear relationship.



$q_o \propto h$ *Linear Valve*

$$q_o = \frac{h}{R} \qquad \text{……………..(1)}$$

Where:

R is related as a linear resistance

If $q_o \alpha\ h^n$ $(n \neq 1)$ Non linear valve

q(t) is a time varying volumetric flowrate with constant density ρ.

Find the T.F. that relates the head to the input flowrate q(t).

We can analyze this system by writing a transient mass balance around the tank:

Mass flow in - mass flow out = rate of accumulation of mass in the tank

$$q\rho - q_o\rho = \frac{dv\rho}{dt} \qquad (v = Ah)$$

$$q - q_o = A\frac{dh}{dt}$$

$$q - \frac{h}{R} = A\frac{dh}{dt} \qquad \text{………………(2)}$$

At staedy state

$$q_s - \frac{h_s}{R} = A\frac{dh_s}{dt} = 0 \qquad \text{………………(3)}$$

Substracting Eq(3) from Eq. (2)

$$(q - q_s) - \frac{h - h_s}{R} = A\frac{d(h - h_s)}{dt}$$

$$let \quad Q = q - q_s \quad , \quad H = h - h_s$$

$$A\frac{dH}{dt}+\frac{H}{R}=Q$$

$AR\frac{dH}{dt}+H=RQ$ Taken laplace for both sides of equation

$(\tau s+1)H(s)=RQ(s)$ where $\tau = AR$

$$\boxed{\frac{H(s)}{Q(s)}=\frac{R}{(\tau s+1)}}$$ First order system equation …………………(4)

$$\boxed{H(s)=\frac{R}{(\tau s+1)}Q(s)}$$ …………………(5)

Comparing the T.F. of the tank with thr T.F. of the thermometer we see that eq. (5) contain the factor (R) which is relate H(t) to Q(t) at s.s. as $s\to 0$, $t\to\infty$

For this reason, a factor R in the transfer function $\frac{R}{\tau s+1}$ is called the steady state gain

To show that

Take $Q(s)=\frac{1}{s}$

Therefore $H(s)=\frac{R}{\tau s+1}.\frac{1}{s}$

Final value theorm

$$\lim_{t\to\infty}H(t)=\lim_{s\to 0}sH(s)=\lim_{s\to 0}s\frac{R}{\tau s+1}.\frac{1}{s}=\lim_{s\to 0}\frac{R}{\tau s+1}=R\ \ s.s\ gain$$

Also to find the level as a function of time

$$H(t)=L^{-1}H(s)$$

3.Mixing Tank

Consider the mixing process shown in Figure in which a stream of solution containing dissolved salt flows at a constant volumetric flow rate *q* into a tank of constant holdup volume *V*. The concentration of the salt in the entering stream *x* (mass of salt/volume) varies with time. It is desired to determine the transfer function relating the outlet concentration *y* to the inlet concentration *x*.

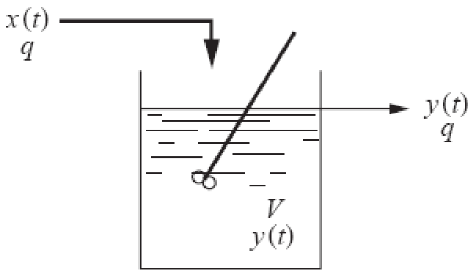


FIGURE 5–4
Mixing process.

If we assume the density of the solution to be constant, the flow rate in must equal the flowrate out, since the holdup volume is fixed.

F: Volumetric flowrate

x, y : Input and output salt concentrations (mass or mole/vol)

Unsteady state material balance

$$Fx - Fy = \frac{dVy}{dt} = V\frac{dy}{dt}$$

Steady state

$$Fx_s - Fy_s = V\frac{dy_s}{dt} = 0$$

Subtracting the above both equations and introducing the deviation variables

$$F(x - x_s) - F(y - y_s) = V\frac{d(y - y_s)}{dt}$$

$$X = x - x_s$$

$$Y = y - y_s$$

$$V\frac{dY}{dt} + FY = FX$$

$$\frac{V}{F}\frac{dY}{dt} + Y = X$$

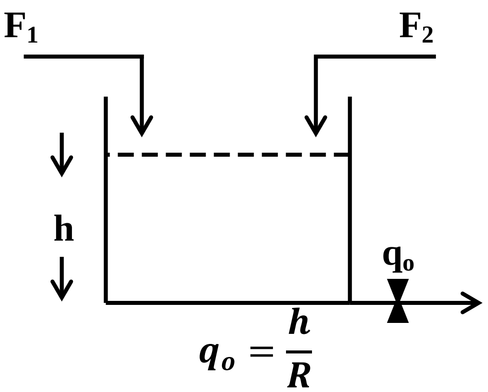
$$\tau\frac{dY}{dt} + Y = X$$

$$(\tau s + 1)Y(s) = X(s)$$

$$\boxed{\frac{Y(s)}{X(s)} = \frac{1}{(\tau s + 1)}}$$ 1st order system , where $\tau = \frac{V}{F}$

<u>Example:</u>

Find the T.F for the system shown in figure



Solution:

$$F_1 + F_2 - \frac{h}{R} = A\frac{dh}{dt}$$

$$F_{1s} + F_{2s} - \frac{h_s}{R} = A\frac{dh_s}{dt}$$

let $\quad \overline{F}_1 = F_1 - F_{1s}$, $\overline{F}_2 = F_2 - F_{2s}$, $\overline{h} = h - h_s$

$$\overline{F}_1 + \overline{F}_2 - \frac{\overline{h}}{R} = A\frac{d\overline{h}}{dt}$$

$$AR\frac{d\overline{h}}{dt} + \overline{h} = \overline{F}_1 R + \overline{F}_2 R$$

$$(\tau s + 1)\overline{h}(s) = \overline{F}_1(s)R + \overline{F}_2(s)R$$

$$\boxed{\overline{h}(s) = \frac{R}{(\tau s + 1)}\overline{F}_1(s) + \frac{R}{(\tau s + 1)}\overline{F}_2(s)}$$

When F_1 constant $\Rightarrow \frac{R}{(\tau s + 1)}\overline{F}_1(s) = 0 \Rightarrow \overline{h}(s) = \frac{R}{(\tau s + 1)}\overline{F}_2(s)$

When F_2 constant $\Rightarrow \frac{R}{(\tau s + 1)}\overline{F}_2(s) = 0 \Rightarrow \overline{h}(s) = \frac{R}{(\tau s + 1)}\overline{F}_1(s)$

4.Heating Tank

Energy balance equation.

$$V\rho C\frac{dT}{dt} = wC(T_i - T) + Q$$

Assumption: constant liquid holdup and constant inflow(w is constant), a linear model result.

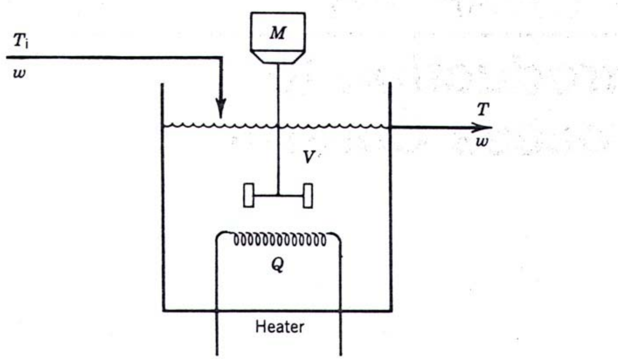


Figure: Continuous stirred-tank heater.

If the process is at steady-state, $dT/dt = 0$

$$0 = wC(T_{is} - T_s) + Q_s$$

Subtract equations

$$V\rho C\frac{dT}{dt} = V\rho C\frac{d(T - T_s)}{dt} = wC[(T_i - T_{is}) - (T - T_s)] + (Q - Q_s)$$

Define some important new variables(Deviation variables).

$$\overline{T} = T - T_s, \overline{T_i} = T_i - T_{is}, \overline{Q} = Q - Q_s$$

By substituting deviation variables for variables.

$$V\rho C\frac{d\overline{T}}{dt} = wC(\overline{T_i} - \overline{T}) + \overline{Q}$$

$$\frac{V\rho}{w}\frac{d\overline{T}}{dt} = \overline{T_i} - \overline{T} + \overline{Q}/wC$$

Let $k = 1/wC, \tau = V\rho/w$

Apply Laplace Transform.

$$\tau s\overline{T}(s) = (\overline{T_i}(s) - \overline{T}(s)) + k\overline{Q}(s)$$

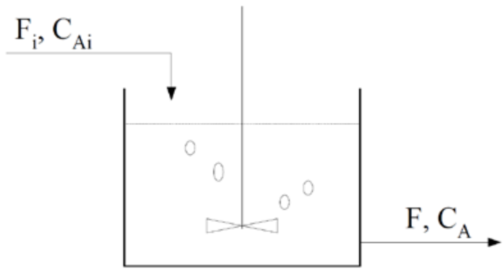
$$(\tau s + 1)\overline{T}(s) = \overline{T_i}(s) + k\overline{Q}(s)$$

$$\boxed{\overline{T}(s) = \frac{1}{\tau s + 1}\overline{T_i}(s) + \frac{k}{\tau s + 1}\overline{Q}(s)}$$

If $\overline{T_i}(s) = 0$ $G_1(s) = \dfrac{\overline{T}(s)}{\overline{Q}(s)} = \dfrac{k}{\tau s + 1}$

If $\overline{Q}(s) = 0$ $G_2(s) = \dfrac{\overline{T}(s)}{\overline{T_i}(s)} = \dfrac{1}{\tau s + 1}$

5. CSTR with constant holdup:



$$V\frac{dC_A}{dt} = FC_{Ai} - FC_A - VKC_A$$

at steady state $V\frac{dC_{A,ss}}{dt} = FC_{Ai,ss} - FC_{A,ss} - VKC_{A,ss}$

by subtracting both equations

$$V\frac{d(C_A - C_{A,ss})}{dt} = F(C_{Ai} - C_{Ai,ss}) - F(C_A - C_{A,ss}) - VK(C_A - C_{A,ss})$$

$$V\frac{d\overline{C}_A}{dt} + (F+VK)\overline{C}_A = F\overline{C}_{Ai} \qquad \div (F+VK)$$

$$\frac{V}{(F+VK)}\frac{d\overline{C}_A}{dt} + \overline{C}_A = \frac{F}{(F+VK)}\overline{C}_{Ai}$$

Let $\tau = \frac{V}{(F+VK)}$ and $K = \frac{F}{(F+VK)}$

$$\tau\frac{d\overline{C}_A}{dt} + \overline{C}_A = K\overline{C}_{Ai}$$

Taking Laplace Transform:

$$(\tau s+1)\overline{C}_A(s) = K\overline{C}_{Ai}(s)$$

$$\boxed{\overline{C}_A(s) = \frac{K\overline{C}_{Ai}(s)}{(\tau s+1)}}$$

Taken sinusoidal change in $\overline{C}_{Ai}(t) \rightarrow \overline{C}_{Ai}(t) = \sin(wt) \rightarrow \overline{C}_{Ai}(s) = \frac{w}{s^2+w^2}$

$$\overline{C}_A(s) = \frac{K}{(\tau s+1)}\frac{w}{s^2+w^2} = \frac{b}{(\tau s+1)} + \frac{cs+d}{s^2+w^2}$$

Multiply both sides by $(\tau s+1)(s^2+w^2)$ gives:

$$Kw = b(s^2+w^2) + (cs+d)(\tau s+1)$$

Let $s = \frac{-1}{\tau} \Rightarrow b = \frac{Kw}{\frac{1}{\tau^2}+w^2} = \frac{Kw\tau^2}{w^2\tau^2+1}$

Let $s = 0 \Rightarrow d = Kw - bw^2 = Kw - \frac{Kw^3\tau^2}{w^2\tau^2+1} = \frac{Kw^3\tau^2 + Kw - Kw^3\tau^2}{w^2\tau^2+1} = \frac{Kw}{w^2\tau^2+1}$

Equating the coefficients of each power of s^2, yields:

$$s^2: \quad b + c\tau = 0 \Rightarrow c = \frac{-b}{\tau} = \frac{-Kw\tau}{w^2\tau^2+1}$$

Substituting the constants in the main equation:

$$\overline{C}_A(s) = \frac{Kw}{w^2\tau^2+1}\left[\frac{\tau^2}{\tau s+1} - \frac{\tau s}{s^2+w^2} + \frac{1}{s^2+w^2}\right] = \frac{Kw}{w^2\tau^2+1}\left[\frac{\tau}{s+1/\tau} - \frac{\tau s}{s^2+w^2} + \frac{1}{s^2+w^2}\right]$$

Using Laplace Table:

$$\boxed{\overline{C}_A(t) = \frac{Kw}{w^2\tau^2+1}\left[\tau e^{\frac{-t}{\tau}} - \tau\cos(wt) + \frac{1}{w}\sin(wt)\right]}$$

Response of first order systems in series

Many physical systems can be represented by several first-order processes connected in series as shown in figure:-

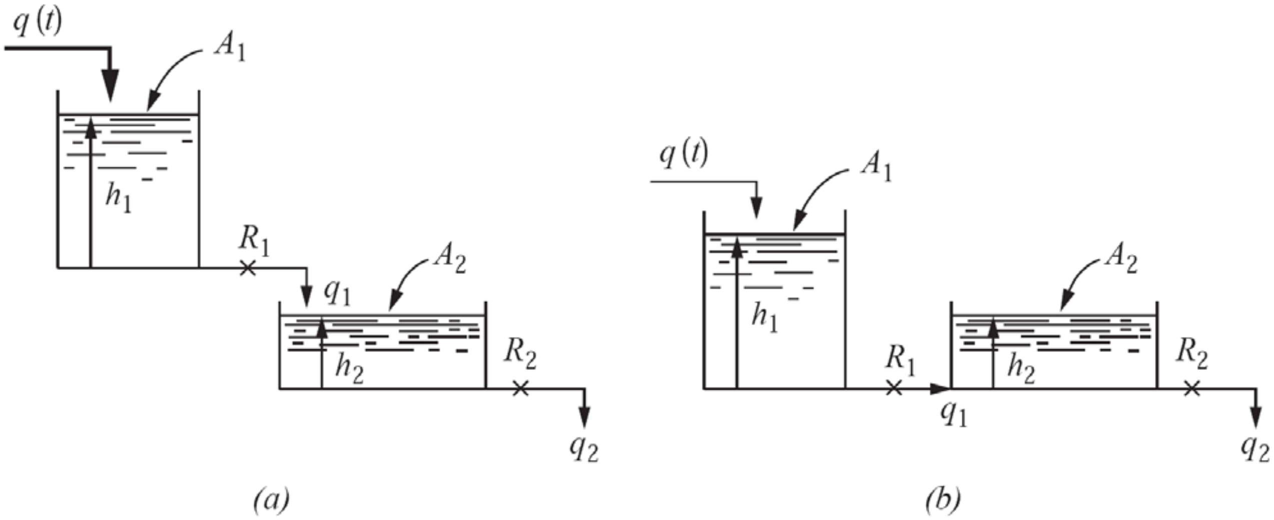
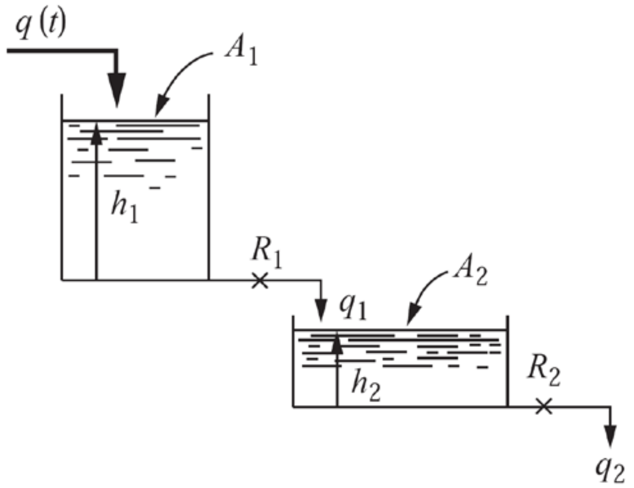


Figure 5.1 Two-tank liquid-level system: *(a)* Non-interacting; *(b)* interacting.

In fig (5.1 a) variation of h_2 does not effect on q_1 then $q_1 = \dfrac{h_1}{R_1}$

In fig (5.1 b) variation of h_2 does effect on q_1 then $q_1 = \dfrac{h_1 - h_2}{R_1}$

1-Non Interacting System



Material balance on tank 1 gives

$$q_i - \frac{h_1}{R_1} = A_1 \frac{dh_1}{dt}$$

At s.s. $q_{is} - \dfrac{h_{1s}}{R_1} = A_1 \dfrac{dh_{1s}}{dt} = 0$

By substracting both equations

$$(q_i - q_{is}) - \frac{h_1 - h_{1s}}{R_1} = A_1 \frac{d(h_1 - h_{1s})}{dt}$$

$$[Q_i - \frac{H_1}{R_1} = A_1 \frac{dH_1}{dt}] \qquad \times R_1$$

$$R_1Q_i = H_1 + R_1A_1\frac{dH_1}{dt}$$

$$\boxed{\frac{H_1(s)}{Q_i(s)} = \frac{R_1}{\tau_1 s+1}}$$ where $\tau_1 = A_1R_1$

Material balance on tank 2 gives

$$\frac{h_1}{R_1} - \frac{h_2}{R_2} = A_2\frac{dh_2}{dt}$$

At s.s. $\frac{h_{1s}}{R_1} - \frac{h_{2s}}{R_2} = A_2\frac{dh_{2s}}{dt} = 0$

By substracting both equations

$$\frac{h_1-h_{1s}}{R_1} - \frac{h_2-h_{2s}}{R_2} = A_2\frac{d(h_2-h_{2s})}{dt}$$

$$\frac{H_1}{R_1} - \frac{H_2}{R_2} = A_2\frac{dH_2}{dt} \qquad \times R_2$$

$$A_2R_2\frac{dH_2}{dt} + H_2 = \frac{R_2}{R_1}H_1$$

$$\tau_2 sH_2(s) + H_2(s) = \frac{R_2}{R_1}H_1(s) \qquad \tau_2 = R_2H_2$$

$$(\tau_2 s+1)H_2(s) = \frac{R_2}{R_1}H_1(s)$$

$H_2(s) = \frac{R_2/R_1}{(\tau_2 s+1)}H_1(s)$ By substituting the lapace transform of H_1(s)

$$H_2(s) = \frac{R_2/R_1}{(\tau_2 s+1)} \times Q_i(s)\frac{R_1}{\tau_1 s+1}$$

$$H_2(s) = \frac{R_2}{(\tau_1 s+1)(\tau_2 s+1)}Q_i(s)$$

$$\boxed{\frac{H_2(s)}{Q_i(s)} = \frac{R_2}{(\tau_1 s+1)(\tau_2 s+1)}}$$ Non-interacting system

In the case of three non-interacting tanks in sereies the transfer function of the system will be as below:-

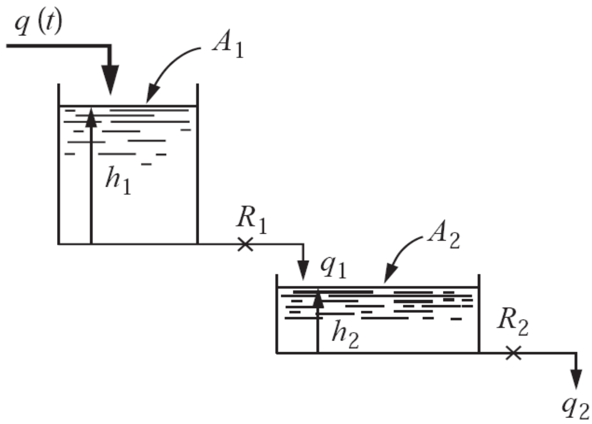
$$\boxed{\frac{H_3(s)}{Q_i(s)} = \frac{R_3}{(\tau_1 s+1)(\tau_2 s+1)(\tau_3 s+1)}}$$

Example:

Two non-interacting tanks are connected in series as shown in Fig. 5.1 *a*. The time constants are $\tau_2 = 1$ and $\tau_1 = 0.5$; $R_2 = 1$. Sketch the response of the level in tank 2 if a unit-step change is made in the inlet flow rate to tank 1.

Solution:

The transfer function for this system is found directly from Equation above thus



$$H_2(s)=\frac{R_2}{(\tau_1 s+1)(\tau_2 s+1)}Q_i(s)$$

Substituting $Q_i(s)=\frac{1}{s}$ Unit step change in Q_i

$$H_2(s)=\frac{R_2}{(\tau_1 s+1)(\tau_2 s+1)}\frac{1}{s}$$

$$=\frac{\alpha_o}{s}+\frac{\alpha_1}{(\tau_1 s+1)}+\frac{\alpha_2}{(\tau_2 s+1)}$$

$$R_2=\alpha_o(\tau_1 s+1)(\tau_2 s+1)+\alpha_1 s(\tau_2 s+1)+\alpha_2 s(\tau_1 s+1)$$

$$let\ s=0 \quad \Rightarrow \quad \alpha_o=R_2$$

$$let\ s=-\frac{1}{\tau_1}\Rightarrow \alpha_1(-\frac{1}{\tau_1})(\tau_2(-\frac{1}{\tau_1})+1)=R_2\Rightarrow \alpha_1(\frac{\tau_2}{\tau_1^2}-\frac{1}{\tau_1})=R_2\Rightarrow \alpha_1(\frac{\tau_2-\tau_1}{\tau_1^2})=R_2$$

$$\therefore \alpha_1=R_2\,(\frac{\tau_1^2}{\tau_2-\tau_1})$$

$$let\ s=-\frac{1}{\tau_2}\Rightarrow \alpha_2(-\frac{1}{\tau_2})(\tau_1(-\frac{1}{\tau_2})+1)=R_2\Rightarrow \alpha_2(\frac{\tau_1}{\tau_2^2}-\frac{1}{\tau_2})=R_2\Rightarrow \alpha_2(\frac{\tau_1-\tau_2}{\tau_2^2})=R_2$$

$$\therefore \alpha_2=R_2\,(\frac{\tau_2^2}{\tau_1-\tau_2})$$

$$H_2(s)=\frac{R_2}{s}+R_2\,(\frac{\tau_1^2}{\tau_2-\tau_1})\frac{1}{(\tau_1 s+1)}+R_2\,(\frac{\tau_2^2}{\tau_1-\tau_2})\frac{1}{(\tau_2 s+1)}$$

$$H_2(s)=R_2[\frac{1}{s}+(\frac{\tau_1\tau_2}{\tau_2-\tau_1})\frac{\tau_1}{\tau_2}\frac{1}{(\tau_1 s+1)}+(\frac{\tau_1\tau_2}{\tau_1-\tau_2})\frac{\tau_2}{\tau_1}\frac{1}{(\tau_2 s+1)}]$$

$$H_2(s)=R_2[\frac{1}{s}-(\frac{\tau_1\tau_2}{\tau_1-\tau_2})\frac{1}{\tau_2}\frac{1}{(s+1/\tau_1)}+(\frac{\tau_1\tau_2}{\tau_1-\tau_2})\frac{1}{\tau_1}\frac{1}{(s+1/\tau_2)}]$$

$$H_2(t)=R_2(1-(\frac{\tau_1\tau_2}{\tau_1-\tau_2})(\frac{1}{\tau_2}e^{-t/\tau_1}-\frac{1}{\tau_1}e^{-t/\tau_2}))$$

$$H_2(t)=1-(\frac{1\times 0.5}{0.5-1})(\frac{1}{1}e^{-t/0.5}-\frac{1}{0.5}e^{-t/1})$$

$$H_2(t) = 1 - \left(\frac{0.5}{-0.5}\right)(e^{-2t} - 2e^{-t})$$

$$\boxed{H_2(t) = 1 + e^{-2t} - 2e^{-t})}$$

$$H_1(s) = \frac{R_1}{\tau_1 s + 1} . Q_i(s)$$

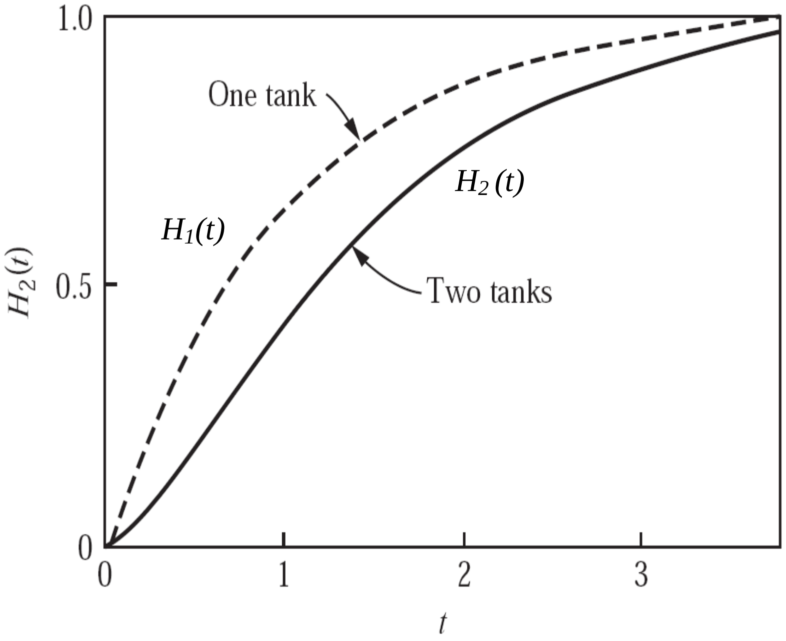
$$H_1(s) = \frac{R_1}{\tau_1 s + 1} . \frac{1}{s}$$

$$H_1(t) = R_1(1 - e^{-t/\tau_1})$$

Substitute $R_1 = 1$

$$H_1(t) = 1(1 - e^{-t/0.5})$$

$$\boxed{H_1(t) = 1 - e^{-2t}}$$



Example:

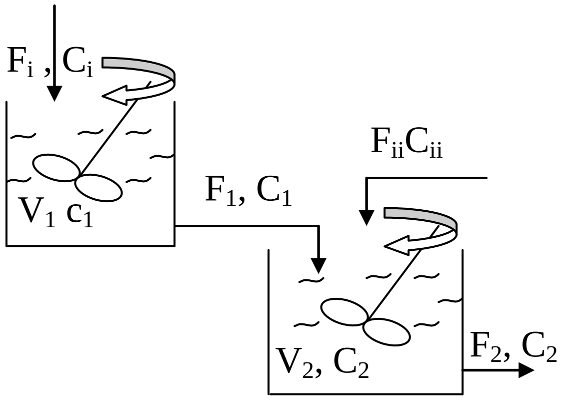
Obtain the transfer function of the following system (no reaction):

Where:

F = volumetric flow rate, $F_i = F_1$

C = conc. of solute in stream

V = liquid volume in tank



(***non-interacting system***)

Solution:

Mass balance on concentration; i.e.

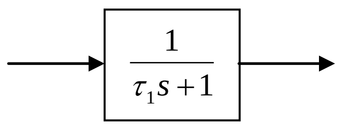
In – out = accumulation

Tank 1: $F_i\, C_i - F_1\, C_1 = V_1 \frac{dC_1}{dt}$

$$\tau_1 \frac{d\overline{C}_1}{dt} + \overline{C}_1 = \overline{C}_i \qquad \text{where } \tau_1 = V_1/F_1$$

Laplace transform $\rightarrow \tau_1\, s\, \overline{C}_1(s) + \overline{C}_1(s) = \overline{C}_i(s)$

$$\boxed{\frac{\overline{C}_1(s)}{\overline{C}_i(s)} = \frac{1}{\tau_1 s + 1}} \quad \text{.... (1)}$$



Tank 2: $F_1 C_1 + F_{ii}\, C_{ii} - F_2\, C_2 = V_2 \frac{dC_2}{dt}$

$$\frac{V_2}{F_2}\frac{d\overline{C}_2}{dt} + \overline{C}_2 = \frac{F_1}{F_2}\overline{C}_1 + \frac{F_{ii}}{F_2}\overline{C}_{ii}$$

$$\tau_2 \frac{d\overline{C}_2}{dt} + \overline{C}_2 = K_1\overline{C}_1 + K_2\overline{C}_{ii}$$

$$\tau_2 = \frac{V_2}{F_2} \quad , \quad K_1 = \frac{F_1}{F_2} \quad , \quad K_2 = \frac{F_{ii}}{F_2}$$

Laplace transform $\rightarrow \tau_2 s \, \overline{C}_2(s) + \overline{C}_2(s) = K_1 \, \overline{C}_1(s) + K_2 \, \overline{C}_{ii}(s)$

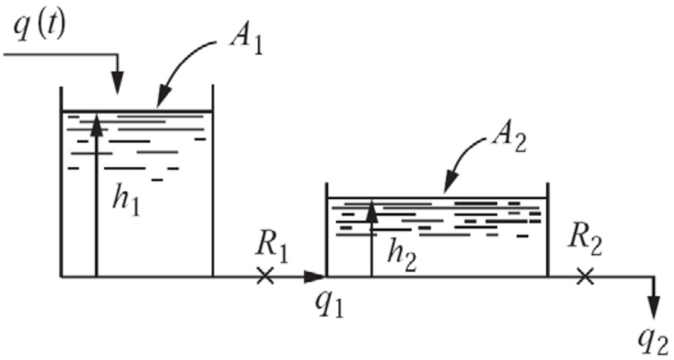
$$\overline{C}_2(s) = \frac{K_1}{\tau_2 s + 1}\overline{C}_1(s) + \frac{K_2}{\tau_2 s + 1}\overline{C}_{ii}(s)$$

Substitute *$C_1(s)$* from Eq. (1)

$$\overline{C}_2(s) = \frac{K_1}{(\tau_1 s + 1)(\tau_2 s + 1)}\overline{C}_i(s) + \frac{K_2}{\tau_2 s + 1}\overline{C}_{ii}(s)$$

2.Interacting System

Material balance on 1st tank



$$q_i - q_1 = A_1 \frac{dh_1}{dt}$$

$$q_i - \frac{h_1 - h_2}{R_1} = A_1 \frac{dh_1}{dt}$$

Steady state

$$q_{is} - \frac{h_{1s} - h_{2s}}{R_1} = A_1 \frac{dh_{1s}}{dt} = 0$$

By substracting both equations

$$(q_i - q_{is}) - \frac{h_1 - h_{1s}}{R_1} + \frac{h_2 - h_{2s}}{R_1} = A_1 \frac{d(h_1 - h_{1s})}{dt}$$

$$[Q_i + \frac{H_2}{R_1} = \frac{H_1}{R_1} + A_1 \frac{dH_1}{dt}] \qquad \times R_1$$

$$Q_i R_1 + H_2 = H_1 + A_1 R_1 \frac{dH_1}{dt}$$

$$\tau_1 \frac{dH_1}{dt} + H_1 = Q_i R_1 + H_2$$

$$(\tau_1 s + 1) H_1(s) = R_1 Q_i(s) + H_2(s)$$

$$H_1(s) = \frac{R_1}{(\tau_1 s + 1)} Q_i(s) + \frac{1}{(\tau_1 s + 1)} H_2(s) \qquad \ldots\ldots\ldots\ldots..(1)$$

Material balance on second tank

$$\frac{h_1 - h_2}{R_1} - \frac{h_2}{R_2} = A_2 \frac{dh_2}{dt}$$

$$\frac{h_{1s} - h_{2s}}{R_1} - \frac{h_{2s}}{R_2} = A_2 \frac{dh_{2s}}{dt} = 0$$

$$[\frac{H_1}{R_1} - \frac{H_2}{R_1} - \frac{H_2}{R_2} = A_2 \frac{dH_2}{dt}] \qquad \times R_2$$

$$A_2 R_2 \frac{dH_2}{dt} + H_2 = \frac{R_2}{R_1}(H_1 - H_2)$$

$$(\tau_2 s + 1) H_2(s) = \frac{R_2}{R_1}(H_1(s) - H_2(s)) \qquad \ldots\ldots\ldots\ldots..(2)$$

Substituting for $H_1(s)$from eq(1) in eq(2)

$$(\tau_2 s + 1) H_2(s) = \frac{R_2}{R_1}[\frac{R_1}{(\tau_1 s + 1)} Q_i(s) + \frac{1}{(\tau_1 s + 1)} H_2(s) - H_2(s)]$$

$$[(\tau_2 s + 1) H_2(s) = \frac{R_2 Q_i(s)}{(\tau_1 s + 1)} + \frac{R_2}{R_1}\frac{H_2(s)}{(\tau_1 s + 1)} - \frac{R_2}{R_1} H_2(s)] \qquad \times (\tau_1 s + 1)$$

$$(\tau_2 s + 1)(\tau_1 s + 1) H_2(s) = R_2 Q_i(s) + \frac{R_2}{R_1} H_2(s) - \frac{(\tau_1 s + 1) R_2}{R_1} H_2(s)]$$

$$(\tau_1 \tau_2 s^2 + \tau_1 s + \tau_2 s + 1) H_2(s) + \frac{\tau_1 R_2 s}{R_1} H_2(s) = R_2 Q_i(s)$$

Let $\frac{\tau_1 R_2}{R_1} = \frac{A_1 R_1 R_2}{R_1} = A_1 R_2 = \tau_{12}$

$$(\tau_1 \tau_2 s^2 + (\tau_1 + \tau_2 + \tau_{12}) s + 1) H_2(s) = R_2 Q_i(s)$$

$$H_2(s) = \frac{R_2}{\tau_1 \tau_2 s^2 + (\tau_1 + \tau_2 + \tau_{12}) s + 1} . Q_i(s)$$ Interacting system

$$H_2(s) = \frac{R_2}{\tau_1 \tau_2 s^2 + (\tau_1 + \tau_2) s + 1} . Q_i(s)$$ Non- Interacting system

The difference between the transfer function for the non-interacting system, and that for the interacting system, is the presence of the cross-product term A_1R_2 in the coefficient of *s*. $\tau_{12} = A_1 R_2$

Example:

To understand the effect of interaction on the transient response of a system, consider a two-tank system for which the time constants are equal ($\tau_1=\tau_2=\tau$).

$\tau_1 = \tau_2 = \tau_{12}=\tau$

$Q_2(t)$=? Output flow rate

$$Q_i(s)=\frac{1}{s}$$

Solution:

Non-interacting system

$$\frac{H_2(s)}{Q_i(s)}=\frac{R_2}{\tau_1\tau_2 s^2+(\tau_1+\tau_2)s+1} \qquad \tau_1=\tau_2$$

$$\frac{H_2(s)}{Q_i(s)}=\frac{R_2}{\tau^2 s^2+2\tau s+1} \quad \text{but } Q_2(s)=\frac{H_2(s)}{R_2}$$

$$\frac{Q_2(s)}{Q_i(s)}=\frac{1}{\tau^2 s^2+2\tau s+1}=\frac{1}{(\tau s+1)(\tau s+1)}=(\frac{1}{\tau s+1})^2$$

If $Q_i(s)=\frac{1}{s}$

$$Q_2(s)=\frac{1}{(\tau s+1)^2}.\frac{1}{s}=\frac{\alpha_o}{s}+\frac{\alpha_1}{(\tau s+1)^2}+\frac{\alpha_2}{\tau s+1}$$

By multiplying both sides by $s(\tau s+1)^2$ and expanding, we get

$$\alpha_o(\tau s+1)^2+\alpha_1 s+\alpha_2 s(\tau s+1)=1$$

$$\alpha_o(\tau^2 s^2+2\tau s+1)+\alpha_1 s+\alpha_2(\tau s^2+s)=1$$

$$s^2(\alpha_o\tau^2+\alpha_2\tau)+s(2\tau\alpha_o+\alpha_1+\alpha_2)+\alpha_o=1$$

$$s^0 \quad \Rightarrow \alpha_o=1$$

$$s^2 \quad \Rightarrow \alpha_o\tau^2+\alpha_2\tau=0 \Rightarrow \tau^2+\alpha_2\tau=0 \Rightarrow \alpha_2=-\tau$$

$$s^1 \quad \Rightarrow 2\alpha_o\tau+\alpha_1+\alpha_2=0 \Rightarrow 2\tau+\alpha_1-\tau=0 \Rightarrow \alpha_1=-\tau$$

$$Q_2(s)=\frac{1}{s}-\frac{\tau}{(\tau s+1)^2}-\frac{\tau}{\tau s+1}$$

$$Q_2(s)=\frac{1}{s}-\frac{\tau}{(\tau s+1)^2}-\frac{\tau}{\tau s+1}$$

$$Q_2(s)=\frac{1}{s}-\frac{1}{\tau}\frac{1}{(s+1/\tau)^2}-\frac{1}{s+1/\tau}$$

$$\boxed{Q_2(t)=1-e^{-t/\tau}-\frac{t}{\tau}e^{-t/\tau}}$$ for non-interacting

Interacting system

If the tanks are interacting, the overall transfer function, according to Equation of interacting system (assuming further that A_1=A_2)

$$Q_2(s)=\frac{1}{\tau^2 s^2+3\tau s+1}.\frac{1}{s}$$

By application of the quadratic formula, the denominator of this transfer function can be written as

$$Q_2(s)=\frac{1}{s}\frac{1}{(0.38\tau s+1)(2.62\tau s+1)}$$

$$Q_2(s)=\frac{\alpha_o}{s}+\frac{\alpha_1}{0.38\tau s+1}+\frac{\alpha_2}{2.62\tau s+1}$$

$$let\ s=0 \Rightarrow \alpha_o=1$$

$$let\ s=-\frac{1}{0.38\tau} \Rightarrow \alpha_1=-0.38\tau\frac{1}{2.62\tau(-\frac{1}{0.38\tau})+1}=0.0664\tau$$

$$let\ s=-\frac{1}{2.62\tau} \Rightarrow \alpha_1=-2.62\tau\frac{1}{0.38\tau(-\frac{1}{2.62\tau})+1}=-3.664\tau$$

$$Q_2(s)=\frac{1}{s}+\frac{0.0664\tau}{0.38\tau s+1}-\frac{3.0664\tau}{2.62\tau s+1}$$

$$Q_2(s)=\frac{1}{s}+\frac{0.0664\tau/0.38\tau}{s+1/0.38\tau}-\frac{3.0664\tau/2.62\tau}{s+1/2.62\tau}$$

$$Q_2(s)=\frac{1}{s}+\frac{0.17}{s+1/0.38\tau}-\frac{1.17}{s+1/2.62\tau}$$

$$\boxed{Q_2(t)=1+0.17e^{-t/0.38\tau}-1.17e^{-t/2.26\tau}}$$

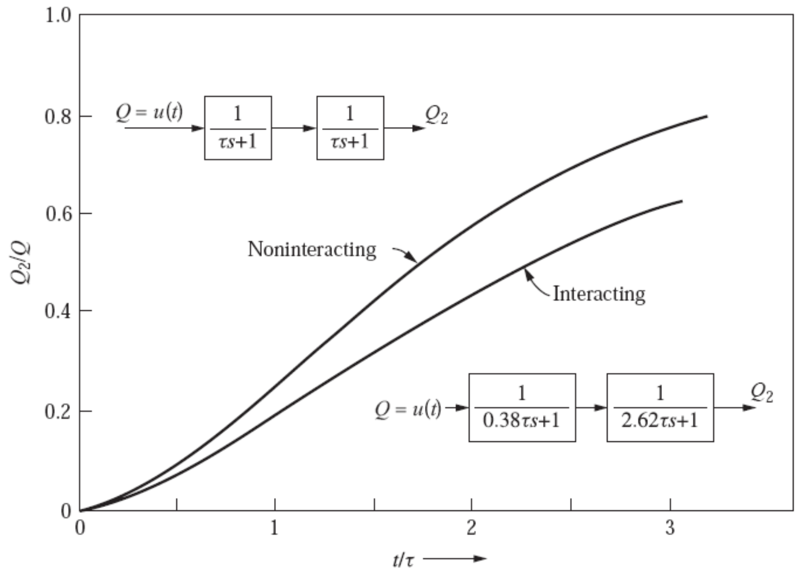


Figure: Effect of interaction on step response of two tank system.

Linearization of non-linear systems

To solve non-linear systems there are two methods:-

1-Linearization method

Making the non-linear function as linear using Taylar series and give approximate results.

2-Non-linear solution

It is difficult and give exact solution

Linearization Technique

$$\left.\begin{aligned} y &= 2t \\ y &= t + x \end{aligned}\right\} \textbf{Linear(all terms to power =1)}$$

$$\left.\begin{aligned} y &= t^2 \\ y &= 2\sqrt{t} \\ y &= ln\,x \end{aligned}\right\} \textbf{Non-Linear (power} \neq \textbf{1)}$$

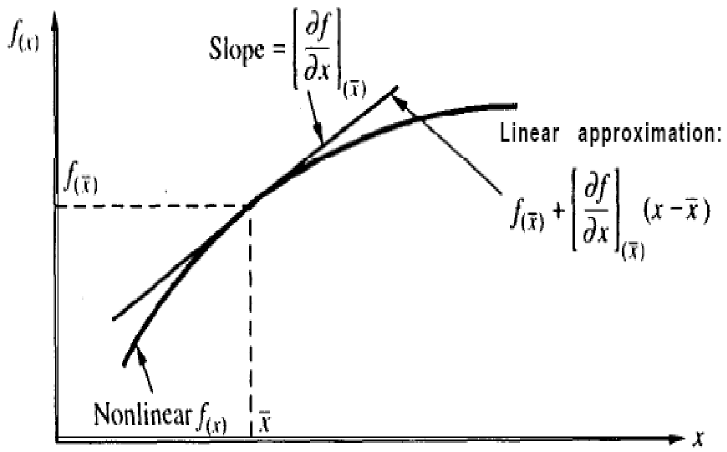
To make the non-linear function linear one use Tayler series.

$$f(x) = f(x_o) + \left.\frac{d\,f(x)}{dx}\right|_{x=x_o}(x - x_o) + \frac{1}{2!}\left.\frac{d^2\,f(x)}{dx^2}\right|_{x=x_o}(x - x_o)^2 + \quad ....$$

Neglacting the non linear terms because their value are very small.

Then

$$\boxed{f(x) = f(x_o) + \left.\frac{d\,f(x)}{dx}\right|_{x=x_o}(x - x_o)}$$



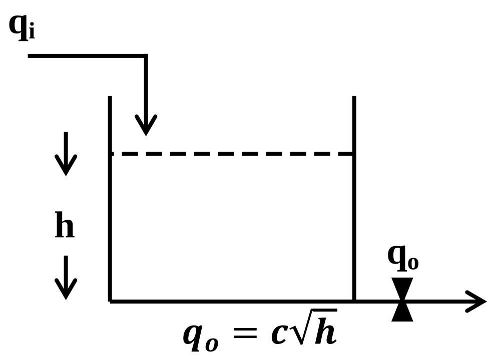
Example:

The flow of water through a valve or other construction usually follow a square-root law.

If $q_o = \frac{h}{R}$ Linear valve

$q_o = c\sqrt{h}$ Non-linear valve

c is a constant



$$q_i\rho - q_o\rho = \frac{d(\rho V)}{dt} = \rho A\frac{dh}{dt}$$

$$q_i - ch^{1/2} = A\frac{dh}{dt} \quad \text{……..(1)}$$

q_o may be expanded around the s.s. value h_s using linearization method

$$h^{1/2} = h_s^{1/2} + \frac{1}{2}\frac{1}{h_s^{1/2}}(h - h_s)$$

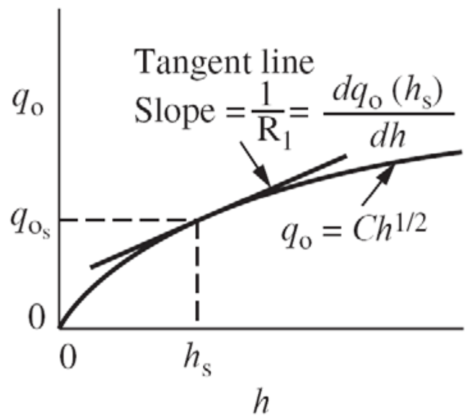
$$q_i - c\left[h_s^{1/2} + \frac{1}{2}\frac{1}{h_s^{1/2}}(h - h_s)\right] = A\frac{dh}{dt} \quad \text{…….. (2)}$$

$$q_{is} - c\left[h_s^{1/2}\right] = A\frac{dh_s}{dt} = 0 \quad \text{at s.s } h=h_s \text{ …….. (3)}$$

$$(q_i - q_{is}) - c\left[\frac{1}{2}\frac{1}{h_s^{1/2}}(h - h_s)\right] = A\frac{d(h - h_s)}{dt}$$

$$Q_i - \left[\frac{c}{2}\frac{1}{h_s^{1/2}}H\right] = A\frac{dH}{dt}$$

$$assume\ \frac{c}{2\sqrt{h_s}} = \frac{1}{R}$$



$$\therefore Q_i - \frac{H}{R} = A\frac{dH}{dt}$$

$$RA\frac{dH}{dt} + H = RQ_i$$

By taking laplace transform

$$(\tau s + 1)H(s) = RQ_i(s)$$

$$\boxed{\frac{H(s)}{Q_i(s)} = \frac{R}{\tau s + 1}} \quad \text{………….. 1st order system}$$

Where $R = \frac{2\sqrt{h_s}}{c}$ and $\tau = AR$

1-Transfer function is similar to linear.

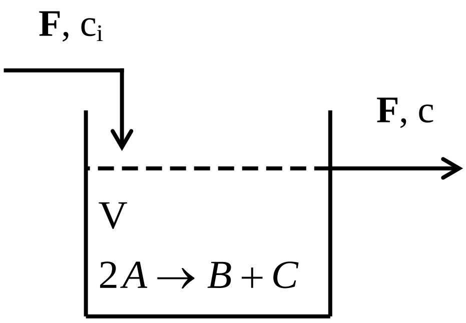
2- R depends on the s.s. condition (at steady state the flow entering the tank equals to the flow leaving the tank,then $q_o = q_{os}$).

Example: Mixing tank with chemical reaction

$2A \rightarrow B + C$

$Reaction\ Rate = r = -kc^2$

$G(s) = \frac{C(s)}{C_i(s)} = ?$



c_i,c : Composition of component (A)

V: Constant=L

F: Constant=L/min

Solution:

In - out - rate of reaction = accumulation

$$Fc_i - Fc - Vkc^2 = V\frac{dc}{dt} \quad \text{Un-steady state}$$

$$c^2 = c_s^2 + 2c_s(c - c_s)$$

$$Fc_i - Fc - Vk[c_s^2 + 2c_s(c - c_s)] = V\frac{dc}{dt}$$

$$Fc_{is} - Fc_s - Vkc_s^2 = V\frac{dc_s}{dt} = 0 \quad \text{Steady state } c = c_s$$

$$F(c_i - c_{is}) - F(c - c_s) - Vk[2c_s(c - c_s)] = V\frac{d(c - c_s)}{dt}$$

$$V\frac{dC}{dt} + (F + 2Vkc_s)C = FC_i \qquad \div (F + 2Vkc_s)$$

$$\tau\frac{dC}{dt} + C = RC_i$$

$$\boxed{\frac{C(s)}{C_i(s)} = \frac{R}{\tau s + 1}} \qquad \text{1st order system}$$

Where $C = c - c_s$, $C_i = c_i - c_{is}$

$$\tau = \frac{V}{F + 2Vkc_s} \quad , \quad R = \frac{F}{F + 2Vkc_s}$$

Linearization of n^th order nonlinear differential equation

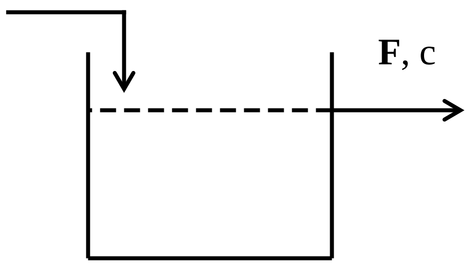
Consider the n^{th} order nonlinear differential equation $f(x_1, x_2 ..., x_n) = u(t)$

Expanding the nonlinear function in a Taylor series about an operating point that x_i^o , $i = 1,2...,n$ satisfies the original differential equation and retaining only the linear terms yields:-

$$f(x_1, x_2, ..., x_n) \approx f(x_1, x_2, ..., x_n)\Big|_{x_i = x_i^0} + \frac{df(x_1, x_2, ...x_n)}{dx_1}\Bigg|_{x_i = x_i^0} (x_1 - x_1^o)$$
$$+ \frac{df(x_1, x_2, ...x_n)}{dx_2}\Bigg|_{x_i = x_i^0} (x_2 - x_2^o) + \qquad + \frac{df(x_1, x_2, ...x_n)}{dx_n}\Bigg|_{x_i = x_i^0} (x_n - x_n^o)$$

Example:

Mixing tank



$$G(s) = \frac{C(s)}{F(s)} = ?$$

c : Variable (kg/L)

F: Variable (L/min)

C_i: Constant

V: Constant

Solution:

$$Fc_i - \underbrace{Fc}_{non\ linear\ term} = V \frac{dc}{dt}$$

$$f(x, y) = f(x_s, y_s) + \frac{\partial f}{\partial x}\Bigg|_{\substack{x = x_s \\ y = y_s}} (x - x_s) + \frac{\partial f}{\partial y}\Bigg|_{\substack{x = x_s \\ y = y_s}} (y - y_s)$$

$$\therefore Fc = F_s c_s + c_s(F - F_s) + F_s(c - c_s)$$

$$Fc_i - [F_s c_s + c_s(F - F_s) + F_s(c - c_s)] = V \frac{dc}{dt} \qquad \text{Un s.s}$$

$$F_s c_i - F_s c_s = V \frac{dc_s}{dt} = 0 \qquad \text{s.s}$$

$$c_i(F - F_s) - c_s(F - F_s) - F_s(c - c_s) = V \frac{d(c - c_s)}{dt}$$

$$c_i X - c_s X - F_s Y = V \frac{dY}{dt}$$

where $Y = c - c_s$
$X = F - F_s$

$$[V\frac{dY}{dt} + F_s Y = (c_i - c_s)X] \qquad \div F_s$$

$$\frac{V}{F_s}\frac{dY}{dt} + Y = \frac{(c_i - c_s)}{F_s}X$$

$$\tau\frac{dY}{dt} + Y = RX$$

Where

$$\tau = \frac{V}{F_s} \qquad , \qquad R = \frac{(c_i - c_s)}{F_s}$$

$$\boxed{\frac{Y(s)}{X(s)} = \frac{R}{\tau s + 1}}$$ 1^{st} order system

Time Delay

The most commonly used model to describe the dynamics of chemical process is First-Order Plus Model Delay Model. By proper choice τ_d , this model can be represent the dynamics of many industrial processes.

- Time delay or dead time between inputs and outputs are very common industrial procsses, engineering systems, economical, and biological systems.

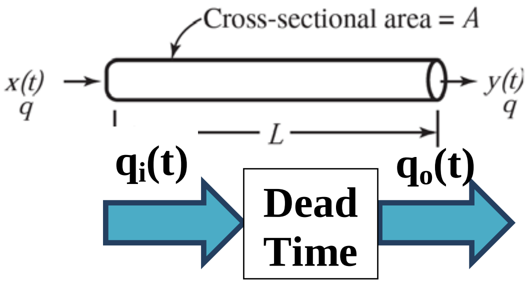
- Transportation and measurement lags, analysis times, computation and communication lags.

Any delay in measuring, in controller action, in actuator operation, in computer computation, and the like, is called transportation delay or dead time, and it always reduces the stability of a system and limits the achievable time of the system.

The Transportation Lag

The transportation lag is the delay between the time an input signal is applied to a system and the time the system reacts to that input signal. Transportation lags are common in industrial applications. They are often called “dead time”.

Dead-Time Approximations:-



$q_i(t)$ = Input to dead-time element.

$q_o(t)$ = Output from dead-time element.

The simplest dead-time approximation can be obtained qraphically or by physical representation.

$$q_o(t) = q_i(t - \tau_d)$$

$$Q_o(s) = Q_i(s) e^{-\tau_d s}$$

$$\frac{Q_o(s)}{Q_i(s)} = e^{-\tau_d s}$$

The accuracy of this approxiamtion depends on the dead time being sufficicently small relative to the rate of the change of the slope of $q_i(t)$. If $q_i(t)$ were a ramp (constant slope), the approximation would be perfect for any value of τ_d. When the slope of $q_i(t)$ varies rapidly, only smal τ_d's will give a good approximation.

If the variation in x(t) were some arbitrary function, as shown in figure below, the response y(t) at the end of the pipe would be identical with x(t) but again delayed by t

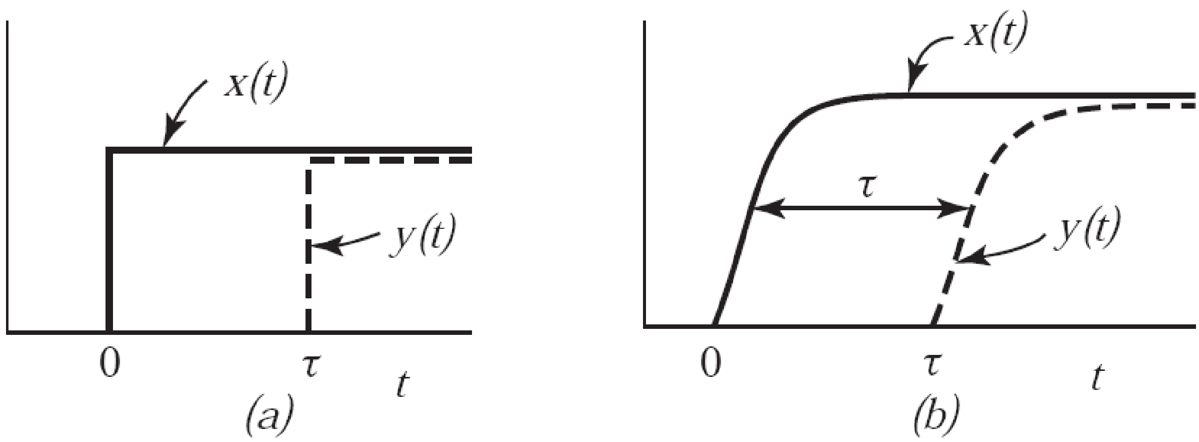
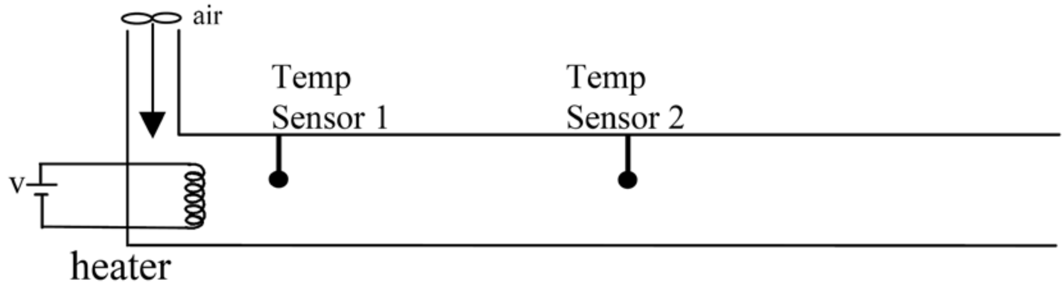


Figure Response of transportation lag to various inputs.

Example: Thermal system



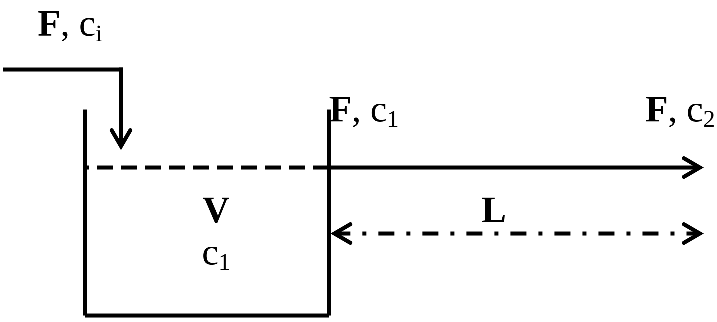
If measured at T_1 this can be modelled as:

$$\frac{T_1(s)}{V(s)} = \frac{K}{\tau s + 1}$$

Due to the delay time the temperature T_2 represented by:

$$\frac{T_2(s)}{V(s)} = \frac{K\, e^{-\tau_d s}}{\tau s + 1}$$

Example: Mixing tank with time delay.



F, V : Constants

Time delay

$$\frac{C_2(s)}{C_1(s)} = e^{-\tau_d s}$$

$$\tau_d = \frac{Volume\ \ of\ \ tube}{Volumetric\ \ flow\, rate} = \frac{AL}{q} = \frac{AL}{uA} = \frac{L}{u}$$

$$\frac{C_1(s)}{C_i(s)} = \frac{R}{\tau s + 1}$$

$$\because \frac{C_2(s)}{C_i(s)} = \frac{C_2(s)}{C_1(s)} \times \frac{C_1(s)}{C_i(s)}$$

$$\therefore \boxed{\frac{C_2(s)}{C_i(s)} = \frac{R}{\tau s + 1} e^{-\tau_d s}}$$ Where $\boxed{\tau_d = \frac{L}{u}}$ (Time Units)

Second order system

A linear second order system under dynamic condition is given by the differential equation:-

$$\frac{1}{\omega_n^2}\frac{d^2Y}{dt^2}+\frac{2\xi}{\omega_n}\frac{dY}{dt}+Y=kX$$

$$\frac{1}{\omega_n}=\tau$$

$$\therefore\ \tau^2\frac{d^2Y}{dt^2}+2\xi\tau\frac{dY}{dt}+Y=kX$$

Where:-

k : Steady state gain

Y : Response value

X : Input disturbing variable

ω_n: Natural frequency of oscillation of the system.

$Y(0)=\overline{Y}(0)=0$

ξ : Damping factor (damping coefficient)

By taking laplace for the above second order differential equation

$$\tau^2 s^2 Y(s)+2\xi\tau s Y(s)+Y(s)=kX(s)$$

$$(\tau^2 s^2+2\xi\tau s+1)Y(s)=kX(s)$$

$$\boxed{G(s)=\frac{Y(s)}{X(s)}=\frac{k}{\tau^2 s^2+2\xi\tau s+1}}$$ T.F. of second order system

If x is sudden force, such as, step change inputs, Y will oscillate depending on the value of damping coefficient ξ .

$\xi<1$ Response will oscillate (Under damped)

$\xi>1$ Response will oscillate (Over damped)

$\xi=1$ Response critical oscillation (critical damped)

Response of second order system

1) Step change response

$$X(s)=\frac{A}{s} \quad \Rightarrow \quad Y(s)=\frac{k}{\tau^2 s^2+2\xi\tau s+1}\times\frac{A}{s}=\frac{k/\tau^2}{s^2+\frac{2\xi}{\tau}s+\frac{1}{\tau^2}}\times\frac{A}{s} \qquad \ldots\ldots(1)$$

The quadratic term in this equation may be factored into two linear terms that contain the roots

$$s_{1,2}=\frac{-\frac{2\xi}{\tau}\pm\sqrt{(\frac{2\xi}{\tau})^2-\frac{4}{\tau^2}}}{2}=\frac{-\xi}{\tau}\pm\frac{\sqrt{\frac{4\xi^2}{\tau^2}-\frac{4}{\tau^2}}}{2}=\frac{-\xi}{\tau}\pm\frac{2\sqrt{\frac{\xi^2-1}{\tau^2}}}{2}$$

$=\frac{-\xi}{\tau}\pm\frac{\sqrt{\xi^2-1}}{\tau}$ =Two real roots

$$s_1=\frac{-\xi}{\tau}-\frac{\sqrt{\xi^2-1}}{\tau} \quad and \quad s_2=\frac{-\xi}{\tau}+\frac{\sqrt{\xi^2-1}}{\tau} \qquad \ldots\ldots(2)$$

Eq. (1) can now be re-written as

$$Y(s)=\frac{kA/\tau^2}{s(s-s_1)(s-s_2)}$$

ξ > 1	**Overdamped**	**Two distinct real roots**
ξ = 1	**Critically Damped**	**Two equal real roots**
0 < ξ < 1	**Underdamped**	**Two complex roots**

$$Y(s)=\frac{k}{\tau^2 s^2+2\xi\tau s+1}\times\frac{A}{s}=\frac{\alpha_0}{s}+\frac{\alpha_1 s+\alpha_2}{\tau^2 s^2+2\xi\tau s+1}$$

$$\alpha_o(\tau^2 s^2+2\xi\tau s+1)+\alpha_1 s^2+\alpha_2 s=kA$$

$s^0 \qquad \alpha_0=kA$

$s^1 \qquad 2\alpha_0\xi\tau+\alpha_2=0 \quad \Rightarrow \quad \alpha_2=-2kA\xi\tau$

$s^2 \qquad \alpha_0\tau^2+\alpha_1=0 \quad \Rightarrow \quad \alpha_1=-kA\tau^2$

$$\therefore Y(s)=kA[\frac{1}{s}-\frac{\tau^2 s+2\xi\tau}{\tau^2 s^2+2\xi\tau s+1}]$$

$$Y(s)=kA[\frac{1}{s}-\frac{s+2\frac{\xi}{\tau}}{(s+2\frac{\xi}{\tau}s+\frac{1}{\tau^2})+\frac{\xi^2}{\tau^2}-\frac{\xi^2}{\tau^2}}]=kA[\frac{1}{s}-\frac{s+2\frac{\xi}{\tau}}{(s+2\frac{\xi}{\tau}s+\frac{\xi^2}{\tau^2})+\frac{1}{\tau^2}-\frac{\xi^2}{\tau^2}}]$$

$$Y(s)=kA[\frac{1}{s}-\frac{s+2\frac{\xi}{\tau}}{(s+\frac{\xi}{\tau})^2+\frac{1-\xi^2}{\tau^2}}]$$

1) **For** $\xi<1$ ==> under damped system

$$Y(s)=kA[\frac{1}{s}-\frac{s+2\frac{\xi}{\tau}}{(s+\frac{\xi}{\tau})^2+(\frac{\sqrt{1-\xi^2}}{\tau})^2}]=kA[\frac{1}{s}-\frac{s+\frac{\xi}{\tau}+\frac{\xi}{\tau}}{(s+\frac{\xi}{\tau})^2+(\frac{\sqrt{1-\xi^2}}{\tau})^2}]$$

$$=kA[\frac{1}{s}-\frac{s+\frac{\xi}{\tau}}{(s+\frac{\xi}{\tau})^2+(\frac{\sqrt{1-\xi^2}}{\tau})^2}-\frac{\frac{\xi}{\tau}}{(s+\frac{\xi}{\tau})^2+(\frac{\sqrt{1-\xi^2}}{\tau})^2}]$$

$$=kA[\frac{1}{s}-\frac{s+\frac{\xi}{\tau}}{(s+\frac{\xi}{\tau})^2+(\frac{\sqrt{1-\xi^2}}{\tau})^2}-\frac{\frac{\xi}{\tau}\times\frac{\tau}{\sqrt{1-\xi^2}}\times\frac{\sqrt{1-\xi^2}}{\tau}}{(s+\frac{\xi}{\tau})^2+(\frac{\sqrt{1-\xi^2}}{\tau})^2}]$$

$$Y(t)=kA[1-e^{(-\xi/\tau)t}\cos\frac{\sqrt{1-\xi^2}}{\tau}t-\frac{\xi}{\sqrt{1-\xi^2}}e^{(-\xi/\tau)t}\sin\frac{\sqrt{1-\xi^2}}{\tau}t]$$

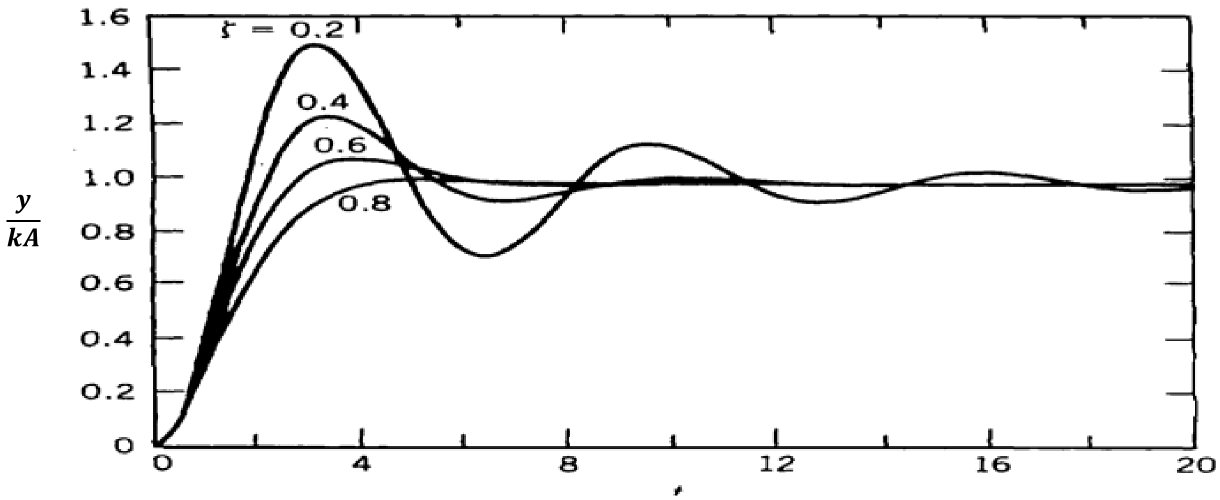
$$w=\frac{\sqrt{1-\xi^2}}{\tau}$$

$$Y(t)=kA[1-e^{(-\xi/\tau)t}(\cos wt+\frac{\xi}{\sqrt{1-\xi^2}}\sin wt]$$

$$r=\sqrt{p^2+q^2}=\sqrt{1+(\frac{\xi}{\sqrt{1-\xi^2}})^2}=\sqrt{\frac{1}{1-\xi^2}}$$

$$\phi=\tan^{-1}\frac{p}{q}=\tan^{-1}\frac{1}{\frac{\xi}{\sqrt{1-\xi^2}}}=\tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$$

$$\boxed{Y(t)=kA[1-e^{(-\xi/\tau)t}(r\sin(wt+\phi))]}$$



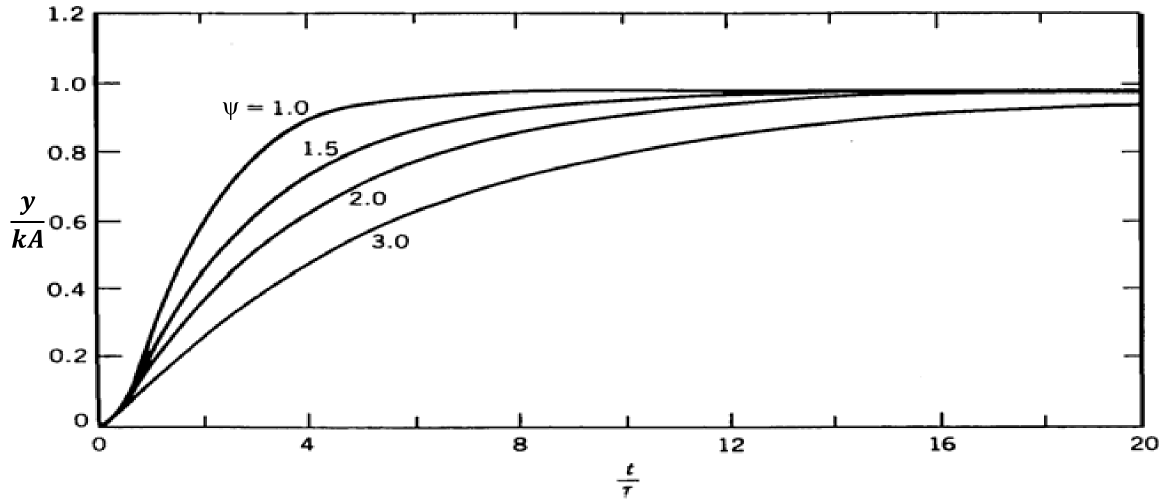
2) For ξ>1 ➔Overdamped system

$$Y(s)=kA\left[\frac{1}{s}-\frac{s+2\frac{\xi}{\tau}}{(s+\frac{\xi}{\tau})^2+\frac{1-\xi^2}{\tau^2}}\right]=kA\left[\frac{1}{s}-\frac{s+\frac{\xi}{\tau}+\frac{\xi}{\tau}}{(s+\frac{\xi}{\tau})^2-\frac{\xi^2-1}{\tau^2}}\right]$$

$$=kA\left[\frac{1}{s}-\frac{s+\frac{\xi}{\tau}}{(s+\frac{\xi}{\tau})^2-(\frac{\sqrt{\xi^2-1}}{\tau})^2}-\frac{\frac{\xi}{\tau}}{(s+\frac{\xi}{\tau})^2-(\frac{\sqrt{\xi^2-1}}{\tau})^2}\right]$$

$$=kA\left[\frac{1}{s}-\frac{s+\frac{\xi}{\tau}}{(s+\frac{\xi}{\tau})^2-(\frac{\sqrt{\xi^2-1}}{\tau})^2}-\frac{\frac{\xi}{\tau}\times\frac{\tau}{\sqrt{\xi^2-1}}\times\frac{\sqrt{\xi^2-1}}{\tau}}{(s+\frac{\xi}{\tau})^2-(\frac{\sqrt{\xi^2-1}}{\tau})^2}\right]$$

$$\boxed{Y(t)=kA[1-e^{-(\xi/\tau)t}(\cosh wt+\frac{\xi}{\sqrt{\xi^2-1}}\sinh wt]}\quad \text{where}\quad w=\frac{\sqrt{\xi^2-1}}{\tau}$$



Terms Used to Describe an Underdamped System

Second order system response for a step change

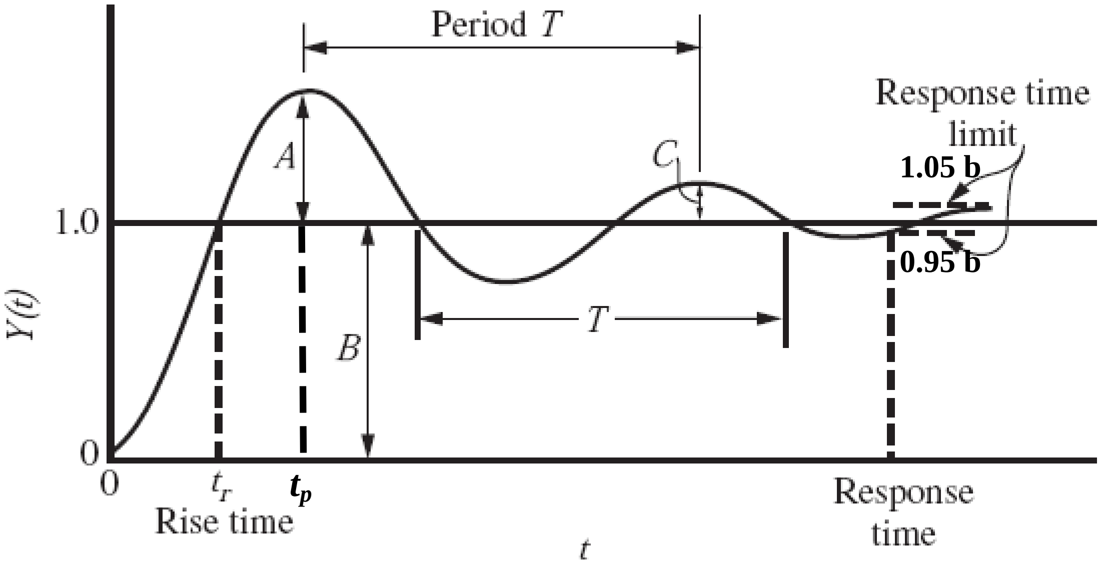


Figure (8.3) Terms used to describe an underdamped second-order response.

***1. Overshoot** (OS)*

Overshoot is a measure of how much the response exceeds the ultimate value (new steady-state value) following a step change and is expressed as the ratio $\frac{A}{B}$ in the Fig. (8-3).

$$OS = \exp\frac{-\pi\xi}{\sqrt{1-\xi^2})}$$

$$OS\,\% = 100 \times OS$$

***2. Decay ratio** (DR)*

The decay ratio is defined as the ratio of the sizes of successive peaks and is given by $\frac{C}{A}$ in Fig. (8.3). where C is the height of the second peak

$$DR = \exp\frac{-2\pi\xi}{\sqrt{1-\xi^2}} = (OS)^2$$

***3. Rise time**(t_r)*

This is the time required for the response to first reach its ultimate value and is labeled in Fig. (8.3).

$$t_r = \frac{\pi - \tan^{-1}\dfrac{\sqrt{1-\xi^2}}{\xi}}{w}$$

4. Response time

This is the time required for the response to come within ±5 percent of its ultimate value and remain there. The response time is indicated in Fig. (8.3).

5. Period of oscillation (T)

The radian frequency (radians/time) is the coefficient of *t* in the sine term; thus,

$$T = \frac{2\pi\tau}{\sqrt{1-\xi^2}}$$

6. Natural period of oscillation

If the damping is eliminated ($\xi=0$), the system oscillates continuously without attenuation in amplitude. Under these "natural" or undamped condition, the radian frequency is $\frac{1}{\tau}$. This frequency is referred to as the natural frequency w_n.

$$w_n = \frac{1}{\tau}$$

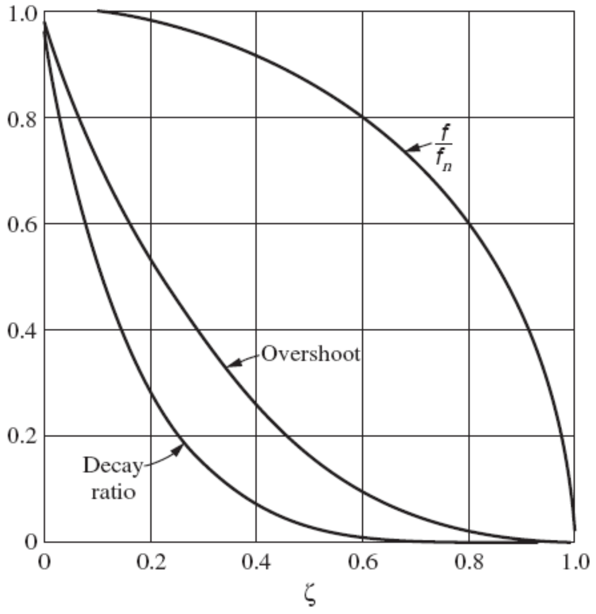
The corresponding natural cyclical frequency f_n and period T_n are related by the expression:-

$$f_n = \frac{1}{T_n} = \frac{1}{2\pi\tau}$$ Thus, τ has the significance of the undamped period.

7- Time to First Peak(t_p) :

Is the time required for the output to reach its first maximum value.

$$t_p = \frac{\pi}{w} = \frac{\pi\tau}{\sqrt{1-\xi^2}}$$



Figure(8.4) Characteristics of a step response of underdamped second-order system.

Derivations

1-Over shoot

$$wt + \phi = \phi + n\pi$$

$$\boxed{t = \frac{n\pi}{w}}$$ max or min n=1, 2, 3 ……

If n=0, 2, 4, 6, ……………. $\therefore$ min

If n=1, 3, 5, 7, ……………. $\therefore$ max

1st max when n=1

$$t = \frac{n\pi}{w} = \frac{\pi}{w}$$

$$y(t) = kA[1 - \frac{1}{\sqrt{1-\xi^2}} e^{\frac{-\xi\pi}{\tau w}} \sin(w\frac{\pi}{w} + \phi)]$$

$$y_{\max} = kA[1 - \frac{1}{\sqrt{1-\xi^2}} e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} (-\sin\phi)]$$

For Underdampded system

$$\cos\phi = -\xi \quad , \quad \sin\phi = \sqrt{1-\xi^2} \quad , \quad \tan\phi = \frac{\sqrt{1-\xi^2}}{-\xi}$$

$$\therefore y_{\max} = kA[1 + \frac{1}{\sqrt{1-\xi^2}} e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} (\sqrt{1-\xi^2})]$$

$$y_{\max} = kA[1 + e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}]$$

$$Overshoot = \frac{A}{B} = \frac{\max - B}{B}$$

$$Overshoot = \frac{kA[1 + e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}] - kA}{kA}$$

$$\boxed{Overshoot = \exp\frac{-\xi\pi}{\sqrt{1-\xi^2)}}}$$

2-Decay Ratio

$\text{Decay ratio} = \frac{C}{A}$ (The ratio of amount above the ultimate value of two sucessive peaks).

$t = \frac{n\pi}{w}$ for n=3 then $t = \frac{3\pi}{w}$

First peak at n=1 $y_{\max} = kA[1 + e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}]$

Second peak at n=3 $y_{\max} = kA[1 + e^{\frac{-3\pi\xi}{\sqrt{1-\xi^2}}}]$

$$Decay\ Ratio = \frac{kA[1 + e^{\frac{-3\pi\xi}{\sqrt{1-\xi^2}}}] - kA}{kA[1 + e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}] - kA} = \frac{e^{\frac{-3\pi\xi}{\sqrt{1-\xi^2}}}}{e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}} = e^{\frac{-2\pi\xi}{\sqrt{1-\xi^2}}}$$

$$\boxed{Decay\ Ratio = \exp\frac{-2\pi\xi}{\sqrt{1-\xi^2}}}$$

<u>3. Rise time</u> (t_r)

It is the time required for the response to first tauch the ultimate line.

$$y(t) = kA[1 - \frac{1}{\sqrt{1-\xi^2}} e^{\frac{-\xi t}{\tau}} \sin(tw + \phi)]$$

At t_r y(t)=kA

$$kA = kA[1 - \frac{1}{\sqrt{1-\xi^2}} e^{\frac{-\xi t}{\tau}} \sin(t_r w + \phi)]$$

$$0 = \sin(t_r w + \phi)]$$

$$tr = \frac{\sin^{-1}(0) - \phi}{w}$$

$$tr = \frac{n\pi - \phi}{w} = \frac{n\pi - \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}{w} =$$

$$\boxed{tr = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}{w}}$$ for n=1

4-Period of oscillation (T)

$$w = Radian\ \ frequency = \frac{\sqrt{1-\xi^2}}{\tau}$$

$$w = 2\pi f \quad also \quad T = \frac{1}{f}$$

$$\boxed{f = \frac{\sqrt{1-\xi^2}}{2\pi\tau}}$$

$$\boxed{\therefore T = \frac{2\pi\tau}{\sqrt{1-\xi^2}}}$$

5. Natural period of oscillation (T_n).

The system free of any damping for ξ=0

$$\text{w, radian of frequency} = \frac{\sqrt{1-\xi^2}}{\tau} \quad \Rightarrow \quad w_n = \frac{1}{\tau} \quad for \quad \xi = 0$$

$$\mathrm{w_n} = 2\pi f_n \quad \Rightarrow \quad \frac{1}{\tau} = 2\pi f_n$$

$$\therefore \boxed{f_n = \frac{1}{2\pi\tau}}$$

6-Response time(t_s)

The time required for the response to reach (±5%) of its ultimate value and remain there.

7- Time to First Peak (t_p)

Is the time required for the output to reach its first maximum value.

$$t = \frac{n\pi}{w}$$

First peak is reached when n=1

$$t_p = \frac{n\pi}{w} = \frac{\pi}{w} = \frac{\pi\tau}{\sqrt{1-\xi^2}}$$

2- Impulse Response

If impulse δ(t) is applied to second order system then transfer response can be written as.

$$Y(s)=\frac{k}{\tau^2 s^2+2\xi\tau s+1}X(s)$$

$$X(s)=Area=A$$

$$Y(s)=\frac{k}{\tau^2 s^2+2\xi\tau s+1}.A$$

$$Y(s)=\frac{kA/\tau^2}{s^2+\frac{2\xi}{\tau}s+\frac{1}{\tau^2}}=\frac{kA/\tau^2}{s^2+\frac{2\xi}{\tau}s+\frac{1}{\tau^2}+(\frac{\xi}{\tau})^2-(\frac{\xi}{\tau})^2}$$

$$=\frac{kA/\tau^2}{s^2+\frac{2\xi}{\tau}s+(\frac{\xi}{\tau})^2+\frac{1}{\tau^2}-(\frac{\xi}{\tau})^2}=\frac{kA/\tau^2}{(s+\frac{\xi}{\tau})^2+\frac{1-\xi^2}{\tau^2}}$$

i) ξ>1

$$Y(s)=\frac{kA/\tau^2}{(s+\frac{\xi}{\tau})^2+\frac{1-\xi^2}{\tau^2}}=\frac{kA/\tau^2}{(s+\frac{\xi}{\tau})^2-\left(\frac{\sqrt{\xi^2-1}}{\tau}\right)^2}=\frac{\frac{kA}{\tau^2}\frac{\tau}{\sqrt{\xi^2-1}}\frac{\sqrt{\xi^2-1}}{\tau}}{(s+\frac{\xi}{\tau})^2-\left(\frac{\sqrt{\xi^2-1}}{\tau}\right)^2}$$

$$Y(t)=\frac{kA}{\tau\sqrt{\xi^2-1}}e^{\frac{-\xi t}{\tau}}\sinh wt$$

$$w=\frac{\sqrt{\xi^2-1}}{\tau}$$

ii)ξ<1

$$Y(s)=\frac{kA/\tau^2}{(s+\frac{\xi}{\tau})^2+\frac{1-\xi^2}{\tau^2}}=\frac{kA/\tau^2}{(s+\frac{\xi}{\tau})^2+\left(\frac{\sqrt{1-\xi^2}}{\tau}\right)^2}=\frac{\frac{kA}{\tau^2}\frac{\tau}{\sqrt{1-\xi^2}}\frac{\sqrt{1-\xi^2}}{\tau}}{(s+\frac{\xi}{\tau})^2+\left(\frac{\sqrt{1-\xi^2}}{\tau}\right)^2}$$

$$Y(t)=\frac{kA}{\tau\sqrt{1-\xi^2}}e^{\frac{-\xi t}{\tau}}\sin wt$$

$$w = \frac{\sqrt{1-\xi^2}}{\tau}$$

iii)ξ=1

$$Y(s) = \frac{kA/\tau^2}{(s+\frac{\xi}{\tau})^2 + \frac{1-\xi^2}{\tau^2}} = \frac{kA/\tau^2}{(s+\frac{1}{\tau})^2 + \frac{1-1^2}{\tau^2}} = \frac{kA/\tau^2}{(s+\frac{1}{\tau})^2}$$

$$Y(t) = \frac{kA}{\tau^2} t e^{-t/\tau}$$

Example A step change from 15 to 31 psi in actual pressure results in the measured response from a pressure indicating element shown in Fig. E5.14.

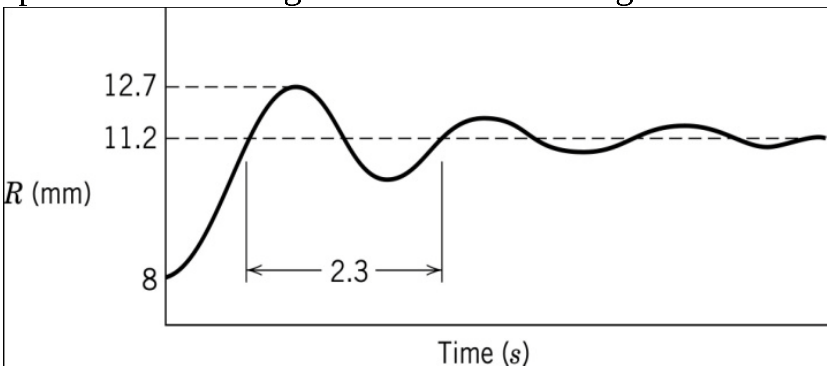


Figure E5.14

Assuming second-order dynamics, calculate all important parameters and write and approximate transfer function in the form

$$\frac{R'(s)}{P'(s)} = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

where *R'* is the instrument output deviation (mm), *P'* is the actual pressure deviation (psi).

Solution:

$$Gain = \frac{11.2\,mm - 8\,mm}{31\,psi - 15\,psi} = 0.20 \;\; mm/psi$$

$$Overshoot = \frac{12.7\,mm - 11.2\,mm}{11.2\,psi - 8\,psi} = 0.47$$

$$Overshoot = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) = 0.47$$

$$\zeta = 0.234$$

$$Period = \frac{2\pi\tau}{\sqrt{1-\zeta^2}} = 2.3\sec$$

$$\tau = 2.3\sec \times \frac{\sqrt{1-0.234^2}}{2\pi} = 0.356\sec$$

$$\frac{R'(s)}{P'(s)} = \frac{0.2}{0.127s^2 + 0.167s + 1}$$

Example: A control system having transfer function is expressed as:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{5}{\tau^2 s^2 + 2\psi\tau s + 1}$$

The radian frequency for the control system is 1.9 rad/min. The time constant is 0.5 min. The control system is subjected to a step change of the magnitude 2.
Calculate :

(i) Rise time

(ii) Decay ratio

(iii) Maximum value of Y(t)

(iv) Response time

Solution:

Given

$$X(s) = \frac{2}{s}$$

Time constant $\tau = 0.5\min$

Radian frequency $w = 1.9\ rad/\min$

$$w = \frac{\sqrt{1-\xi^2}}{\tau} \quad \Rightarrow \quad 1.9 = \frac{\sqrt{1-\xi^2}}{0.5} \quad \Rightarrow \quad \xi = 0.312$$

i) Rise time

$$tr = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}{\mathrm{w}} = \frac{3.1416 - \tan^{-1}\frac{\sqrt{1-0.312^2}}{0.312}}{1.9} = 1.0\min$$

ii) Deacay ratio $= \frac{C}{A} = \exp\left(\frac{-2\pi\xi}{\sqrt{1-\xi^2}}\right) = \exp\left(\frac{-2\pi 0.312}{\sqrt{1-0.312^2}}\right)$

$$\therefore Decay\ ratio = 0.127$$

iii) Ultimate value of the response $\mathrm{Y_{ultimate}}\ (\mathrm{B})$ at $\mathrm{t} \to \infty$

$$\frac{Y(s)}{X(s)} = \frac{5}{0.25s^2 + 0.316s + 1}$$

$$X(s) = \frac{2}{s}$$

$$Y(s) = \frac{10}{s(0.25s^2 + 0.316s + 1)}$$

$$\lim_{t\to\infty} Y(t) = \lim_{s\to 0}[sY(s)] = \lim_{s\to 0} \frac{10}{(0.25s^2 + 0.316s + 1)} = 10$$

$$Y_{ultimate}(B) = 10$$

Maximum value of response $= B(1+\frac{B}{A})$

Overshoot= $\frac{B}{A} = \exp(\frac{-\pi\xi}{\sqrt{1-\xi^2}})$

Decay ratio=Overshoot^2

0.127=Overshoot^2

overshoot = 0.356 = $\frac{B}{A}$

Maximum value of response =10(1+0.356)=13.56

iv) Response time ts =3 $\frac{\tau}{\xi}$ =4.8077 min for $\pm 5\%$ *of ultimate value*

The control system

A liquid stream at a temperature T_i, enters an insulated, well-stirred tank at a constant flow rate w (mass/time). It is desired to maintain (or control) the temperature in the tank at T_R by means of the controller. If the indicated (measured) tank temperature T_m differs from the desired temperature T_R, the controller senses the difference or ***error,*** $E = T_R - T_m$

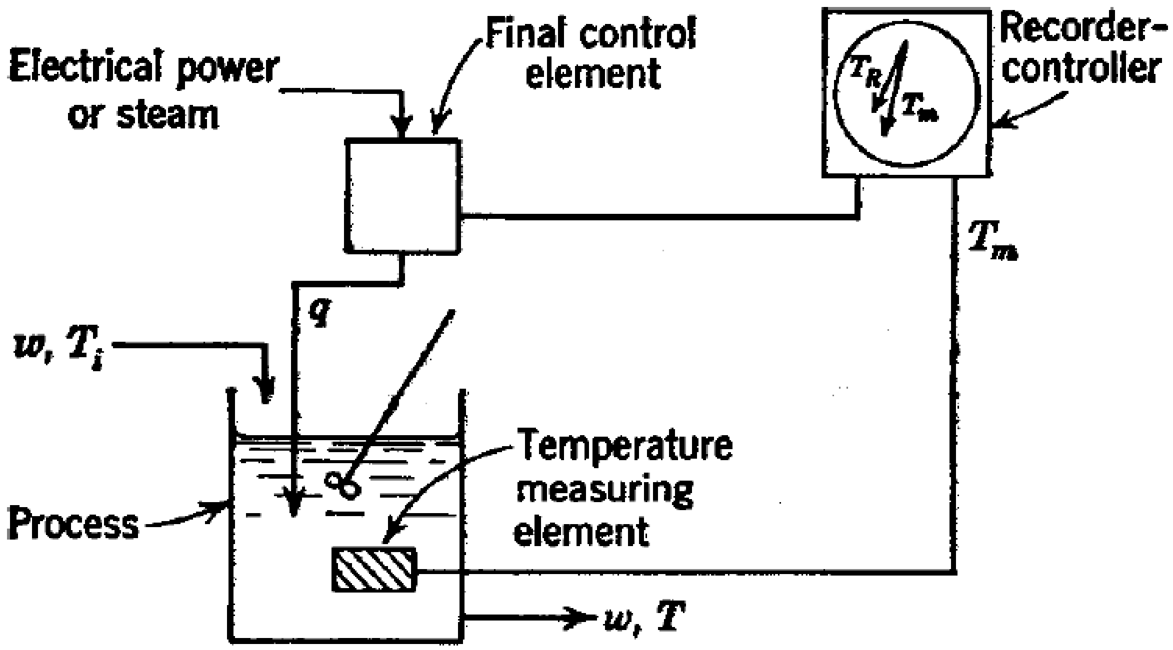


Figure (8.1) Control system for a stirred-tank heater.

There are two types of the control system:-

1) **Negative feedback control system**

   Negative feedback ensures that the difference between T_R and T_m is used to adjust the control element so that tendency is to reduce the error.

   $E=T_R-T_m$

2) **Positve feedback control system**

   If the signal to the compartos were obtained by adding T_R and T_m we would have a positive feedback systems which is inherently unstable. To see that this is true, again assume that be system is at steady state and that $T=T_R=T_i$.

   If T_i were to increase, T and T_m would increas which would cause the signal from the compartor to increase, with the result that the heat to the system would increse.

   At s.s. $T=T_R=T_{in}$

   $E=T_R+T_m$

Servo Problem versus Requlator Problem

❖ **Servo Problem**

There is no change in load T_i, and that we are interested in changing the bath temperature (change in the desired value (set point) with no disturbance load).

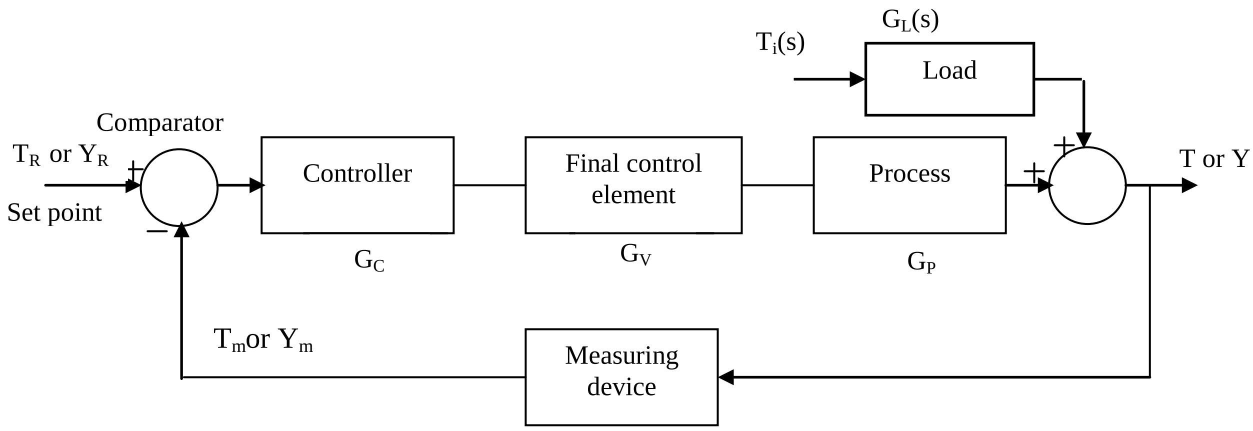
❖ **Requlating problem**

The desired value T_R is to remain fixed and the purpose of the control system is to maintain the controlled variable T_R in spite of change in load if there is a change in the input variable (disturbance load).

Control system elements

Control system elements are:-

1) Process
2) Measuring element
3) Controller
4) Final Control Element



Closed Loop Feedback control

Development of block Diagram

Process

The procedure for developing the transfer function remain the same.
An unsteady-state energy balance around the heating tank gives.

$$W\,Cp(T_i - T_o) + q - W\,Cp(T - T_o) = \rho CpV\frac{dT}{dt}$$

Where T_o is the reference temperature

At steady state, $\frac{dT}{dt} = 0$

$$W\,Cp(T_{is} - T_o) + q_s - W\,Cp(T_s - T_o) = \rho CpV\frac{dT}{dt} = 0$$

By substracting both equations

$$W\,Cp((T_i - T_{is}) - (T - T_s)) + q - q_s = \rho CpV\frac{d(T - T_s)}{dt}$$

Note that the refernece temperature T_o cancels in the subtraction. If we introduce thedeviation variables.

$$\overline{T}_i = T_i - T_{is}$$
$$\overline{T} = T - T_s$$
$$Q = q - q_s$$

$$W\,Cp(\overline{T}_i - \overline{T}) + Q = \rho CpV\frac{d\overline{T}}{dt}$$

Taking the laplace transform gives

$$W\,Cp(\overline{T}_i(s) - \overline{T}(s)) + Q(s) = \rho CpVs\overline{T} \qquad \div W\,Cp$$

$$\frac{\rho V}{W}s\overline{T} + \overline{T}(s) = \frac{Q(s)}{WCp} + \overline{T}_i(s)$$

The last expression can be written as

$$\boxed{\overline{T}(s) = \frac{1}{(\tau s + 1)}\frac{Q(s)}{WCp} + \frac{\overline{T}_i(s)}{\tau s + 1}}$$

Where

$$\tau = \frac{\rho V}{W}$$

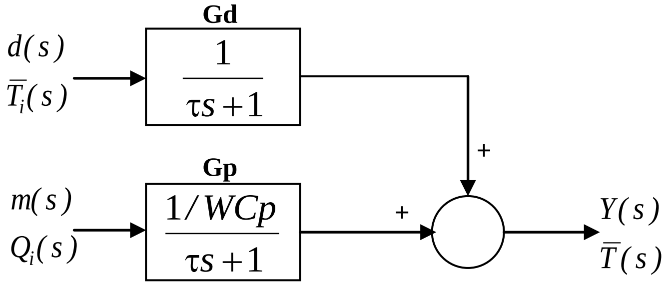
$\overline{T}(s)$ or $Y(s)$ = controlled variable
$Q(s)$ or $m(s)$ = manipulated variable
$\overline{T}_i(s)$ or $d(s)$ = disturbance variable

If there is a change in Q(t) only then $\overline{T}_i(t) = 0$ and the transfer function relating $\overline{T}_i$ to Q is

$$\frac{\overline{T}(s)}{Q(s)} = \frac{1}{(\tau s+1)}\frac{1}{WCp}$$

If there is a change in $\overline{T}_i(s)$ only then Q(t)=0 and the transfer function relating $\overline{T}$ to $\overline{T}_i$ is

$$\frac{\overline{T}(s)}{\overline{T}_i(s)} = \frac{1}{(\tau s+1)}$$



Block Diagram for process

$$Y(s) = G_p \cdot m(s) + G_d \cdot d(s)$$

Measuring Element

The T.F. of the temperature-measuring element is a first order system

$$\frac{\overline{T}_m(s)}{\overline{T}(s)} = \frac{k_m}{\tau_m s+1} \qquad \Rightarrow \overline{T}_m(s) = G_m \overline{T}(s)$$

$$G_m = \frac{k_m}{\tau_m s+1}$$

Where $\overline{T}$ and $\overline{T}_m$ are deviation variables defined as

$\overline{T} = T - T_s$

$\overline{T}_m = T_m - T_{ms}$

$$K_m = steady\ state\ gain = \frac{\Delta Output}{\Delta input}$$

τ_m=time lag (time constant)=(1-9) sec

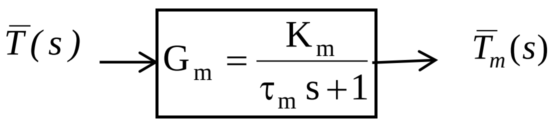
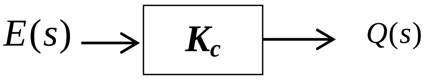


Figure Block diagram of measuring element

Controller and final control element

The relationship for proportional controller is

$$\frac{P(s)}{G(s)} = G_c(s)$$



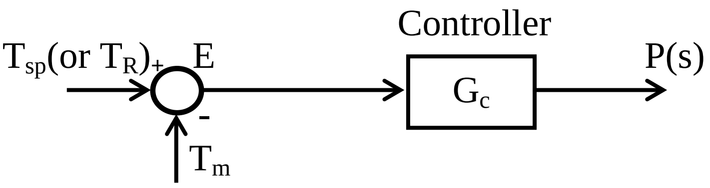
$$Q(s) = K_C E(s)$$

$$P = \overline{P} - \overline{P}_s$$

$$E = \overline{T}_R - \overline{T}_m$$

G(s) for propertional controller $G_c(s) = K_C$

$\overline{T}_R = \overline{T}_m = \overline{T}$ at steady state

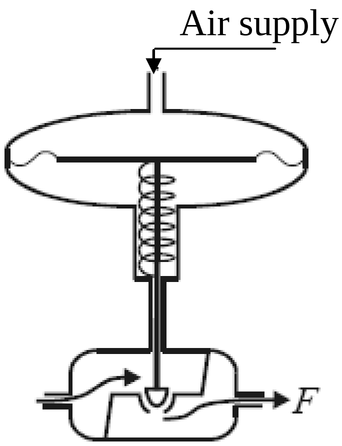


Final control Elements:

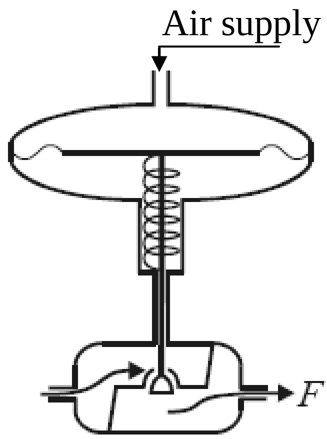
Control valve, Heater, Variec, Motor, pump, damper, louver, …. etc.

Control valve

Control valve that can control the rate of flow of a fluid in proportion to the amplitude of a pressure (electrical) signal from the controller. From experiments conducted on pneumatic valves, the relationship between flow and valve-top pressure for a linear valve can often be represented by a first-order transfer function:



Control valve (Air to close)



Control valve (Air to open)

Figure Pneumatic control valve (air-to-close).

(air to close)

open → max flow rate; close → flow rate = 0

$$P = (3-15)\text{ psig}$$

(air to open)

close → flow rate = 0; open → max flow rate

$$P = (3-15)\text{ psig}$$

Transfer Function of Control Valve

$$Gv(s) = \frac{m(s)}{p(s)} = \frac{Q(s)}{P(s)} = \frac{K_V}{\tau_v s + 1}$$

K_v=steady state gain $= \dfrac{\Delta\text{ Output}}{\Delta\text{ Input}} = \left(\dfrac{Q_2 - Q_1}{P_2 - P_1}\right)_{S.S}$

$\tau_v = \text{Time lag}$

$\tau_v \le 10\text{ sec}$ (Good)

Where:

K_v: steady-state gain i.e., the constant of proportionality between steady-state flow rate and valve-top pressure.

τ_v: time constant of the valve and is very small compared with the time constants of other components of the control system. A typical pneumatic valve has a time constant of the order of 1 sec. Many industrial processes behave as first-order systems or as a series of first-order systems having time constants that may range from a minute to an hour. So the lag of the valve is negligible and the T. F. of the valve sometimes is approximated by:

$$\frac{Q(s)}{P(s)} = K_V$$

The time constant of lag valve depends on the size of valve, air supply characteristics, whether a valve positioner is used, etc.

Control Action

It is the manner, in which the automatic controller compares the actual value of the process output with the actual desired value, determines the deviations and produce a control signal which will reduce the deviation to zero or to small value.

Classification of industrial automatic controller:

They are classified according to their control action as:

1) On-off controller
2) Proportional controller (P)
3) Integral controller (I)
4) Proportional plus Integral controller (PI)
5) Proportional plus Derivative controller (PD)
6) Proportional plus Integral plus Derivative controller (PID)

The automatic controller may be classified according to the kind of power employed in the operation, such as pneumatic controller, hydraulic controller or electronic controller.

Self operated controller: In this controller the measuring element (sensor) and the actuator in one unit. It is widely used for the water and gas pressure control.

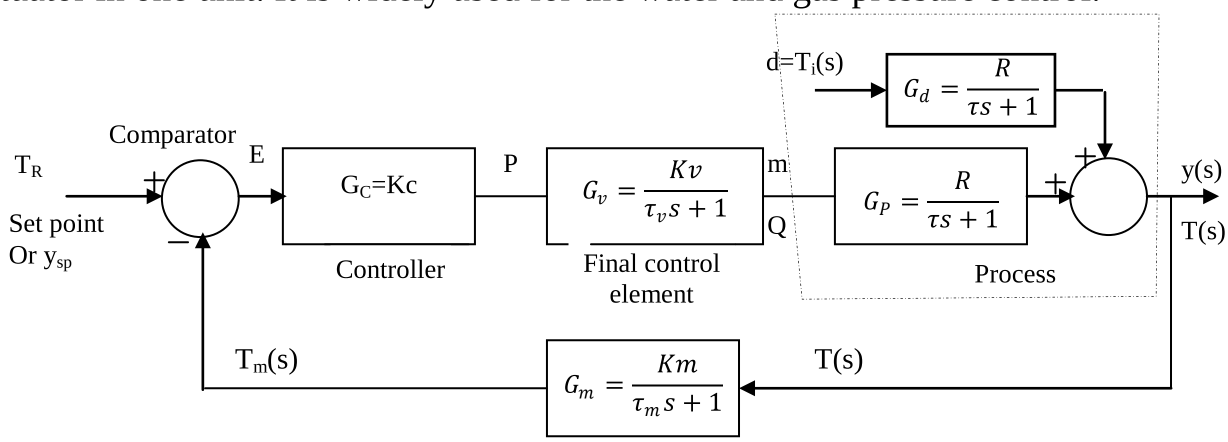


Figure: Closed loop block diagram of first order system

Types of Feedback Controllers

1) Proportional controller (P):

For a controller with a proportional control action, the relationship between the output of the controller, p(t), and the actuating error signal (input to controller) is

$$P(t) = K_C \times E(t)$$

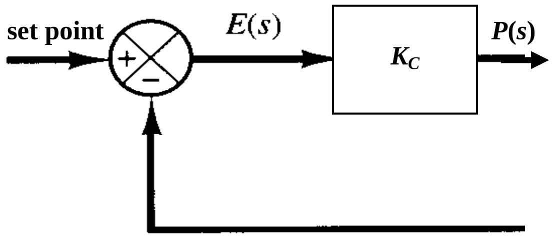
$$p(t) - p_s = K_C \times E(t)$$

$$p(t) = K_C \times E(t) + p_s$$

$$G_C = K_C = \frac{P(s)}{E(s)} \qquad (T.F)$$

$$K_C = (\frac{\Delta p}{\Delta E})_{s.s}$$

$$\boxed{P(s) = K_C \times E(s)}$$



Proportional Band (Band Width)

Is defined as the error (expressed as a percentage of the range of measured variable) required to make the valve from fully close to fully open.

$$\text{P.B} = \frac{1}{\text{K}_\text{C}} \times 100\ \%$$

On-Off Control

On-Off control is a special case of proportional control.

If the gain K_C is made very high, the valve will move from one extreme position to the other if the set point is slightly changed. So the valve is either fully open or fully closed (The valve acts like a switch).

The P.B. of the on-off controller reaches a zero because the gain is very high

$$\text{P.B} \approx 0$$

2) Propertional-Integral controller (PI):

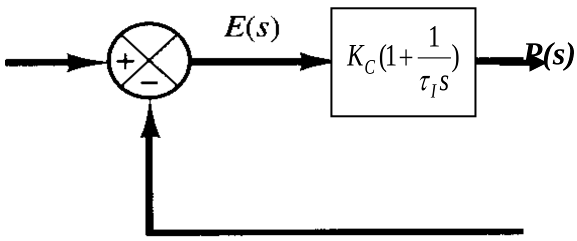
This mode of control is described by the relationship

$$p(t) = p_s + K_C[E(t) + \frac{1}{\tau_I}\int_0^t E(t)dt]$$

K_C : Steady state gain

τ_I : Integral time constant

$$(p(t) - p_s) = P(t) = K_C E(t) + \frac{K_C}{\tau_I}\int_0^t E(t)dt$$



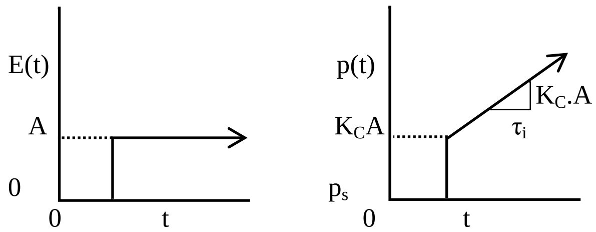
Taking L.T

$$\frac{P(s)}{E(s)} = K_C(1+\frac{1}{\tau_I s}) = G_c(s)$$

Prob(10-1): PI controller with step change in error $E(s) = \frac{A}{s}$

$$P(s) = K_C(1+\frac{1}{\tau_I s})\frac{A}{s}$$

$$\therefore P(t) = K_C A + \frac{K_C A}{\tau_I} t$$



Response of a PI controller (lineaer)

Y=c+mX

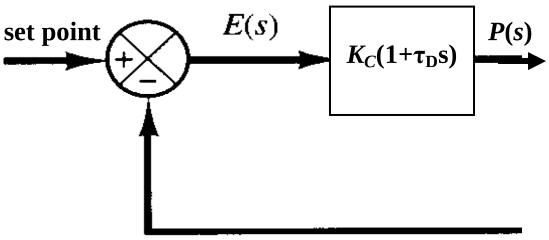
t	E(t)	P(t)

3) Proportional-derivative control (PD):

$$p(t) = K_C[\,E(t) + \tau_D \frac{dE(t)}{dt}] + p_s$$

$$(p(t) - p_s) = P(t) = K_C E(t) + K_C \tau_D \frac{dE(t)}{dt}$$

$$\frac{P(s)}{E(s)} = K_C + K_C \tau_D s$$



$$\frac{P(s)}{E(s)} = K_C(1+\tau_D s) = G_c$$

K_C: gain

τ_D : Derivative time (rate time)

Example:

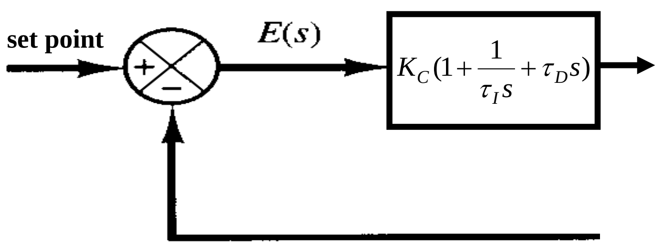
For Ramp Error $E(t) = At$ (Ramp) $E(s) = \frac{A}{s^2}$

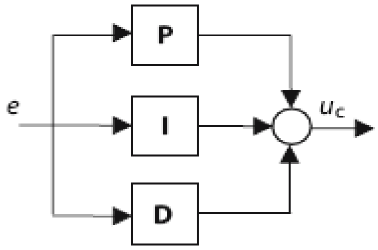
$$P(s) = K_C(1+\tau_D s)\times E(s) = K_C(1+\tau_D s)\times\frac{A}{s^2} = \frac{AK_C}{s^2} + \frac{K_C A \tau_D}{s}$$

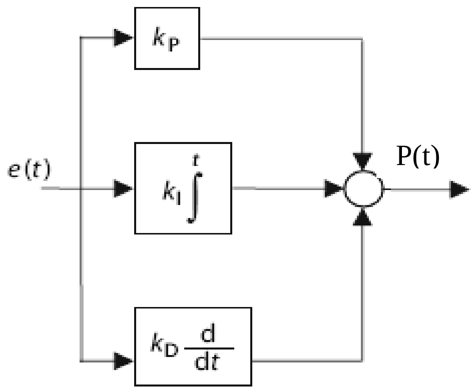
$$P(t) = K_C At + K_C A\tau_D$$

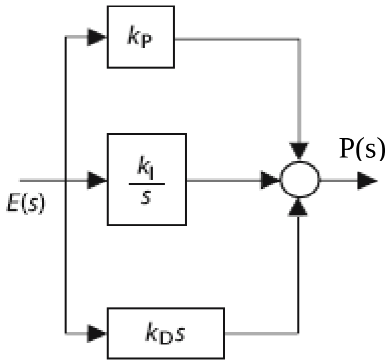
4) Proportional-Integral-Derivative (PID) controller

$$p(t) = K_C[E(t) + \frac{K_C}{\tau_I}\int_0^t E(t)dt + \tau_d \frac{dE(t)}{dt}] + p_s$$









$$\frac{P(s)}{E(s)} = K_C(1 + \frac{1}{\tau_I s} + \tau_D s)$$

Motivation for Addition of Integral and Derivative Control Modes

The value of the controlled variable is seen to rise at time zero owing to the disturbance. With no control, this variable continues to rise to a new steady-state value.

- ❖ With control, after some time the control system begins to take action to try to maintain the controlled variable close to the value that existed before the disturbance occurred.
- ❖ With proportional action only, the control system is able to arrest the rise of the controlled variable and ultimately bring it to rest at a new steady-state value. The difference between this new steady-state value and the original value (the set point, in this case) is called *offset.*
- ❖ The addition of integral action eliminates the offset; the controlled variable ultimately returns to the original value. This advantage of integral action is balanced by the disadvantage of a more oscillatory behavior.
- ❖ The addition of derivative action to the PI action gives a definite improvement in the response. The rise of the controlled variable is arrested more quickly, and it is returned rapidly to the original value with little or no oscillation.

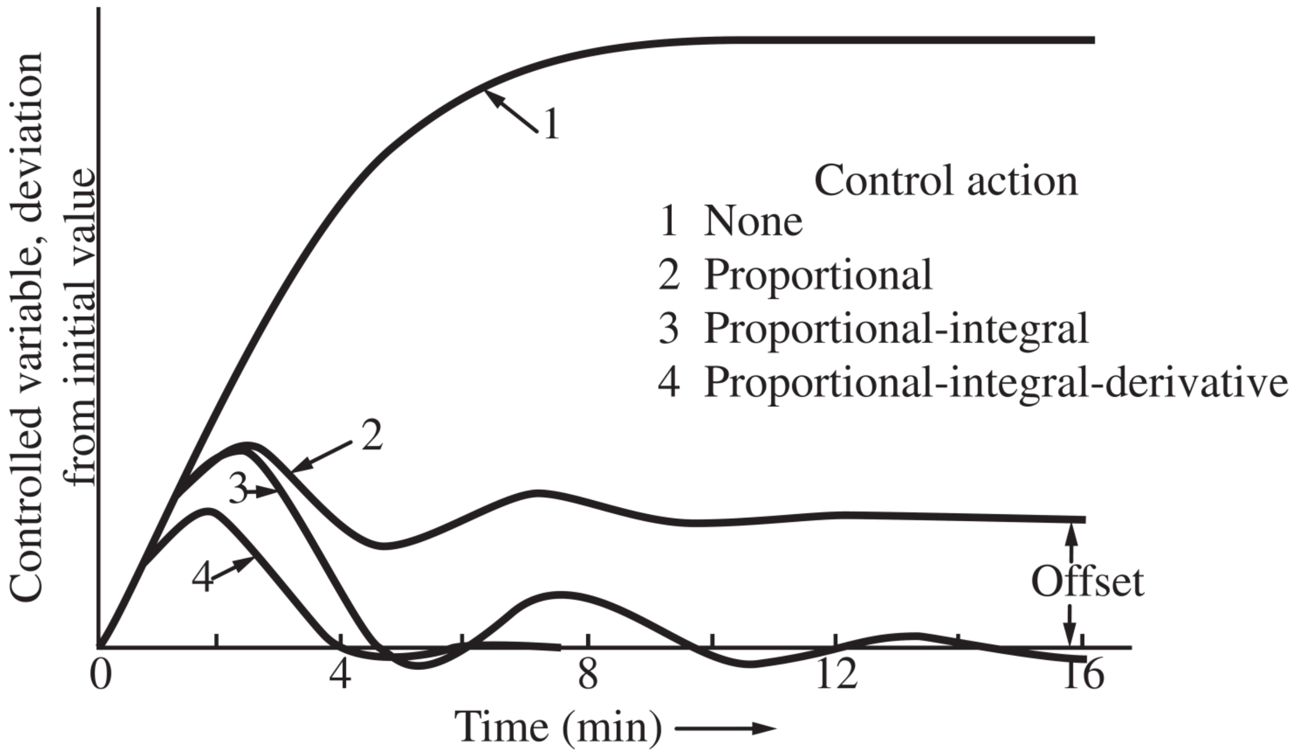


Figure: Response of a typical control system showing the effects of various modes of control

Example: A unit-step change in error is introduced into a PID controller. If K_c = 10, τ_I = 1, and τ_D = 0.5, plot the response of the controller, m(t).

Solution:

The equation of PID controller is

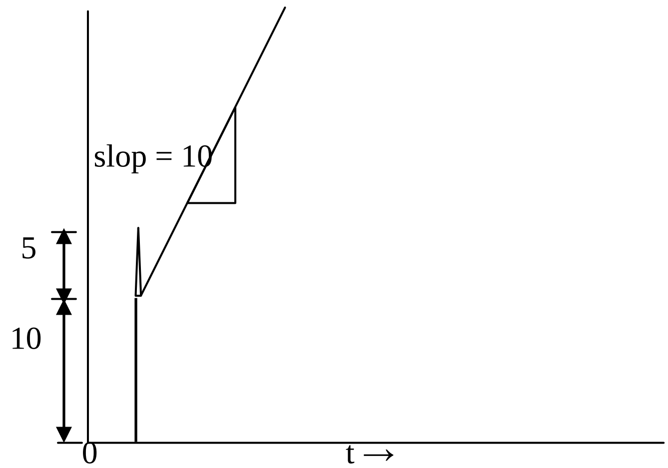
$$\frac{P(s)}{E(s)} = K_C\left(1+\frac{1}{\tau_I s}+\tau_D s\right)\ m(t)$$

$$E(s)=\frac{1}{s}$$

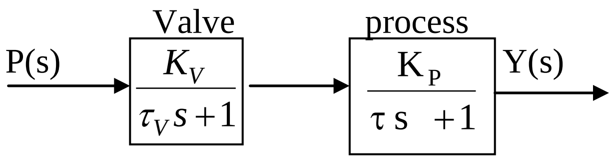
$$P(s)=\frac{10}{s}\left(1+\frac{1}{s}+0.5s\right)$$

$$P(s)=\frac{10}{s}+\frac{10}{s^2}+5$$

$$m(t)=10+10t+5\delta(t)$$



Example: Consider the 1st order T. F. of the process with control valve



If we assume no interaction;
The T. F. from P(s) to Y(s) is

$$\frac{Y(s)}{P(s)} = \frac{K_V K_p}{(\tau_v s+1)(\tau\, s+1)}$$ For a unit step input in P

Y(s) = $\frac{1}{s}\frac{K_V K_p}{(\tau_v s+1)(\tau s+1)}$

y(t) = $K_v K_p \left[1 - \frac{\tau_v \tau}{\tau_v - \tau}\left(\frac{1}{\tau}e^{-t/\tau_v} - \frac{1}{\tau_v}e^{-t/\tau}\right)\right]$

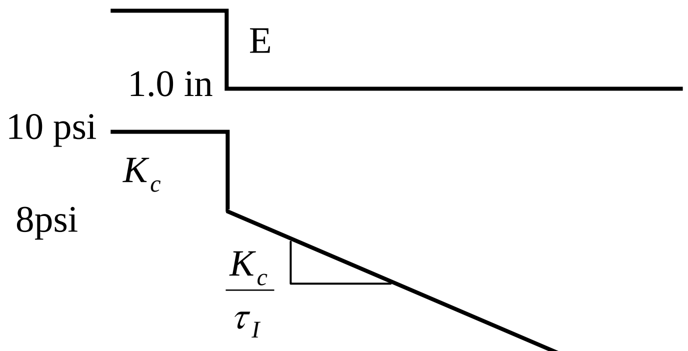
If $\tau >> \tau_v$ then the T. F. is $\frac{Y(s)}{P(s)} = \frac{K_v K_p}{(\tau s+1)}$

For a unit step input in p
y(t) = $K_v K_p (1 - e^{-t/\tau})$

Example: a pneumatic PI controller has an output pressure of 10 psi when the set point and pen point are togather. The set point is suddenly displaced by 1.0 in (i.e a step change in error is introduced) and the follwing data are obtained.

Time (s)	0-	0+	20	60	80
Psi	10	8	7	5	3.5

Determine the actual gain (psi/inch displacement) and the integral time



For PI control

$$p(t) = K_c + \frac{K_c}{\tau_I}\int E dt + p_s$$

For E=1

$$p(t) = K_c + \frac{K_c}{\tau_I} t + p_s$$

From the above figure

$K_c = 2$

$$\frac{K_c}{\tau_I} = \frac{7-5}{60-20} = \frac{2}{40}$$

$\tau_I = 20 K_c = 20 \times (2) = 40 \sec$

Example: (A) a unit-step change in error is introduced into a pid controller, if $K_c=10$, $\tau_I=1$ and $\tau_D=0.5$ plot the response of the controller P(t).
(B) if the error changed with a ratio of 0.5 in/min plot the response of p(t).

Solution:

(A) For a PID control

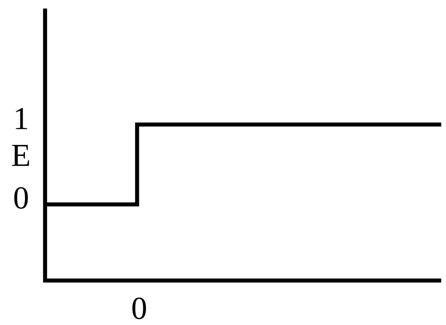
$$p(t)=K_c t+\frac{K_c}{\tau_I}\int Edt+K_c\tau_D\frac{dE}{dt}+p_s$$

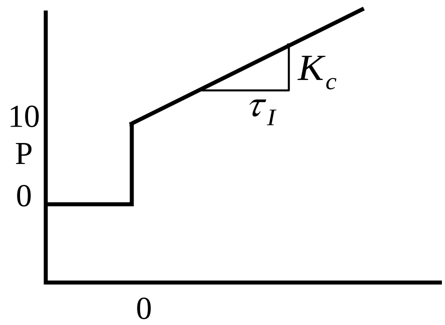
For a unit step change in error E(t)=1

At t=0 $\quad p(0)=K_c+p_s$

t>0 $\quad p(t)=K_c+\frac{K_c}{\tau_I}t+p_s$

$$P=p-p_s=10+10t$$



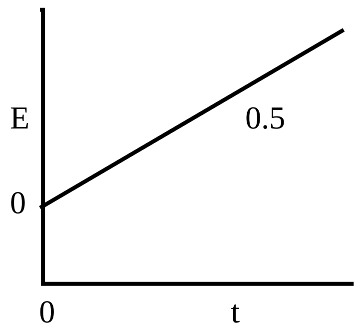


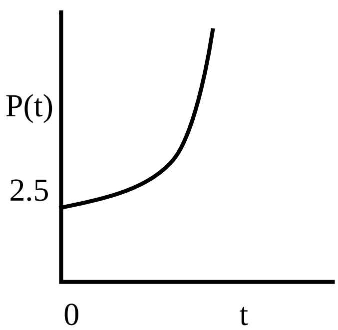
(B) E=0.5 t $\quad \frac{dE}{dt}=0.5$ and $\int dEdt=\int 0.5dt$

$$\therefore\ p(t)=10\times 0.5t+10\int 0.5tdt+10\times 0.5\times 0.5+p_s$$

$$p(t)-p_s=5t+2.5t^2+2.5$$

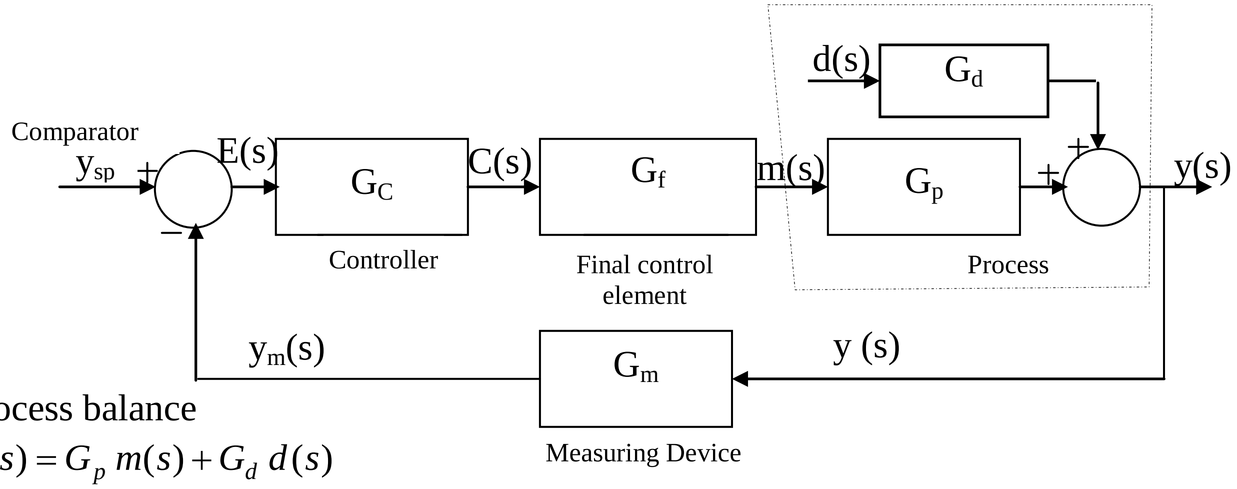
$$\mathrm{P(t)=p(t)-p_s=2.5+5t+2.5t^2}$$





t	P(t)
0	2.5
1	10
2	22.5
3	40
4	62.5
5	90

Overall transfer function of a closed- loop control system:



Process balance

$y(s) = G_p\, m(s) + G_d\, d(s)$

Measuring device

$y_m(s) = G_m\, y(s)$

Controller system

$E(s) = y_{sp}(s) - y_m(s)$ Comparator

$C(s) = G_c\, E(s)$ Controller

Final control element

$m(s) = G_f\, C(s)$

Algebra manipulation of the above equations and arrange then

$y(s) = G_p\, m(s) + G_d\, d(s)$

$y(s) = G_p\, G_f C(s) + G_d\, d(s)$

$y(s) = G_p\, G_f G_c E(s) + G_d\, d(s)$

$y(s) = G_p\, G_f G_c\, (y_{sp}(s) - y_m(s)) + G_d\, d(s)$

$y(s) = G_p\, G_f G_c\, (y_{sp}(s) - G_m\, y(s)) + G_d\, d(s)$

$y(s) = G_p\, G_f G_c\, y_{sp}(s) - G_p\, G_f G_c G_m\, y(s) + G_d\, d(s)$

$(1 + G_c\, G_f G_p G_m)\, y(s) = G_c\, G_f G_p\, y_{sp}(s) + G_d\, d(s)$

$$y(s) = \frac{G_c G_f G_p}{1 + G_c G_f G_p G_m}\, y_{SP}(s) + \frac{G_d}{1 + G_c G_f G_p G_m}\, d(s)$$

Let $\mathbf{G = G_C\, G_f\, G_P}$

$$\therefore\, y(s) = \frac{G}{1 + GG_m}\, y_{SP}(s) + \frac{G_d}{1 + GG_m}\, d(s)$$

$$\frac{G}{1 + GG_m} = G_{SP} \qquad \frac{G_d}{1 + GG_m} = G_{load}$$

Types of control proplems:

1) **Servo systems**:

The distubance does not change (i.e. $\bar{d}(s)=0$) while the set point undergoes change. The feedback controller act in such away as to keep y close to the changing $\bar{y}_{sp}$. The T.F. of closed loop system of this type is:

$$\bar{y}(s)=\frac{G_p G_f G_c}{1+G_c G_f G_p G_m}\bar{y}_{SP}(s)$$

2) **Regulated systems**:

In these systems the set point (desired value) is constant ($\bar{y}_{sp}(s)=0$) and the change occurring in the load. The T.F. of closed loop control system of this type is:

$$\bar{y}(s)=\frac{G_d}{1+G_c G_f G_p G_m}\bar{d}(s)$$

$$\bar{y}(s)=G_{load}\,\bar{d}(s)$$

The feedback controller tries to eliminate the impact of the load change ***d*** to keep ***y*** at the desired setpoint.

Effect of controllers on the response of a controlled process:

(1) Effect of Propertional Control

The general T.F of the closed loop controller is:

$$\bar{y}(s)=\frac{G_c G_f G_p}{1+G_c G_f G_p G_m}\bar{y}_{SP}(s)+\frac{G_d}{1+G_c G_f G_p G_m}\bar{d}(s) \qquad (*)$$

Consider $G_m=1$, $G_f=1$

Also for propertional controller $G_C=K_C$

And eqn. (*) becomes

$$\bar{y}(s)=\frac{K_c G_p}{1+K_c G_p}\bar{y}_{SP}(s)+\frac{G_d}{1+K_c G_p}\bar{d}(s) \qquad (**)$$

For a first order systems

$$\tau_p\frac{dy}{dt}+y=K_p m+K_d d$$

Which gives

$$\bar{y}(s)=\frac{K_p}{\tau_p s+1}\bar{m}(s)+\frac{K_d}{\tau_p s+1}\bar{d}(s)$$

Thus for the uncontrolled system we have time constant= τ_p

Static gains: K_p for manipolation and K_d for load

put $G_p=\frac{K_p}{\tau_p s+1}$ and $G_d=\frac{K_d}{\tau_p s+1}$

Then by substitution in eqn. (**) and take the closed loop reponse as

$$\bar{y}(s)=\frac{K_c\frac{K_p}{\tau_p s+1}}{1+K_c\frac{K_p}{\tau_p s+1}}\bar{y}_{SP}(s)+\frac{\frac{K_d}{\tau_p s+1}}{1+K_c\frac{K_p}{\tau_p s+1}}\bar{d}(s)$$

$$\bar{y}(s)=[\frac{K_P K_C}{\tau_p s+1+K_p K_C}\bar{y}_{SP}(s)+\frac{K_d}{\tau_p s+1+K_p K_C}\bar{d}(s)]\qquad\times\frac{1+K_p K_C}{1+K_p K_C}$$

$$\bar{y}(s)=\frac{\frac{K_P K_C}{1+K_p K_C}}{\frac{\tau_p s}{1+K_p K_C}+\frac{1+K_p K_C}{1+K_p K_C}}\bar{y}_{SP}(s)+\frac{\frac{K_d}{1+K_p K_C}}{\frac{\tau_p s}{1+K_p K_C}+\frac{1+K_p K_C}{1+K_p K_C}}\bar{d}(s)$$

Rearrange the last eqn.

$$\boxed{\bar{y}(s)=\frac{\bar{K}_P}{\bar{\tau}_p s+1}\bar{y}_{SP}(s)+\frac{\bar{K}_d}{\bar{\tau}_p s+1}\bar{d}(s)}$$

Where

$\bar{\tau}_p=\frac{\tau_P}{1+K_p K_C}$ Closed loop time constant

$\bar{K}_p=\frac{K_p K_C}{1+K_p K_C}$ Closed loop ststic gain

$\bar{K}_d=\frac{K_d}{1+K_p K_C}$ Closed loop ststic gain

The close-loop response has the follwing charactrstics:-

1- It remains first order with respect to load and setpoint change
2- The time constant has been reduced $(\bar{\tau}_p<\tau_P)$ which mean that the closed-loop response has become faster than the open loop response, to change in set point or load.
3- The static gain have been decreased.

Disadvantage of Propertional control

Consider a servo problem with a unit step in the set point

$\bar{y}_{sp} = \frac{1}{s} \qquad d(s) = 0$

$\bar{y} = \frac{\bar{K}_p}{\tau_p s + 1} \cdot \frac{1}{s}$

$\bar{y}(t) = \bar{K}_p (1 - e^{-t/\tau_p})$

$\therefore \bar{y}(\infty) = \bar{K}_p$

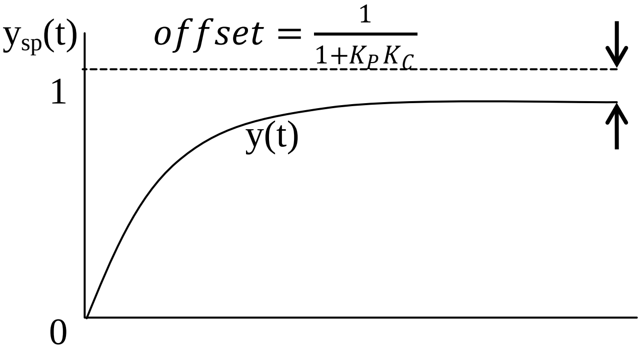
The ultimate response of $t \to \infty$ never reaches the desired new setpoint. There is always a discrepancy called ***offset*** which is equal to:

Offset = New set point - Ultimate value

$= 1 - \bar{K}_p = 1 - \frac{K_p K_C}{1 + K_p K_C}$

$\therefore offset = \frac{1}{1 + K_p K_C}$

Offset decreases as K_C becomes larger an thoretically $offset \to 0 \quad when \quad K_C \to \infty$



2- Effect of Integral Control

Consider a servo problem, $\bar{d}(s) = 0$

$$\bar{y}(s) = \frac{G_c G_f G_p}{1 + G_c G_f G_p G_m} \bar{y}_{SP}(s) \qquad (*)$$

Consider $G_m = G_f = 1$

For the 1st order process $G_p = \frac{k_p}{\tau_p s + 1}$

For a simple integral control

$G_c = K_c \frac{1}{\tau_I s}$

Sub in eqn. (*)

$$\bar{y}(s)=\frac{\frac{K_P}{\tau_p s+1}\cdot\frac{K_C}{\tau_I s}}{1+\frac{K_P}{\tau_p s+1}\cdot\frac{K_C}{\tau_I s}}\bar{y}_{SP}(s)=\frac{K_P K_C}{(\tau_p s+1)(\tau_I s)+K_P K_C}\bar{y}_{SP}(s)$$

$$\bar{y}(s)=\frac{\frac{K_P K_C}{K_P K_C}}{\frac{\tau_p\tau_I s^2}{K_P K_C}+\frac{\tau_I s}{K_P K_C}+\frac{K_P K_C}{K_P K_C}}\bar{y}_{SP}(s)$$

$$\bar{y}(s)=\frac{1}{\tau^2 s^2+2\psi\tau s+1}\bar{y}_{SP}(s) \qquad (**)$$

Where

$$\tau=\sqrt{\frac{\tau_I\tau_p}{K_p K_c}} \qquad \psi=\frac{1}{2}\sqrt{\frac{\tau_I}{\tau_p K_p K_c}}$$

Eqn. (**) indicates an important effect of the integral control action:-

1- It increases the order of the dynamic for the closed-loop reponse.
Thus for a first-order uncontrolled process, the response of the closed-loop becomes second order.

2- Increase K_C decreases ψ $\therefore$ more oscillatory

3- To examine the effect of integral on s.s error

$$\bar{y}(s)=\frac{1}{\tau^2 s^2+2\psi\tau s+1}\bar{y}_{SP}(s)$$

If $\bar{y}_{SP}(s)=\frac{1}{s}$

The ultimate value= AK=1*1=1

$\therefore$ offset= New setpoint-ultimate value

=1-1=0

It indicate that the integral control eliminates any offset

3- Effect of Derivative Control Action

For derivative control

$G_c=K_c\tau_D s$

$$\bar{y}(s)=\frac{\frac{K_P}{\tau_p s+1}\cdot K_c\tau_D s}{1+\frac{K_P}{\tau_p s+1}\cdot K_c\tau_D s}\bar{y}_{SP}(s)=\frac{K_P K_c\tau_D s}{\tau_p s+1+K_P K_c\tau_D s}\bar{y}_{SP}(s)$$

$$\bar{y}(s)=\frac{K_P K_c\tau_D s}{(\tau_p+K_p K_c\tau_D)s+1}\bar{y}_{SP}(s) \qquad (*)$$

Eqn. (*) indicates that:-

1- The derivative control does not change the order of the reponse.

2- The effective time constant of the closed-loop response $(\tau_p + K_p K_C \tau_D > \tau_p)$

This means that the response of the controlled process is slower than that of the original first-order process and as K_c increase the response become slower.

Effect of Composite Control Action

1- Effect of PI control

Combination of propertional and integral control modes lead to the follwing effects on the response of closed-loop system.

1- The order of the response inceases (effect of I mode).

2- The offset is eliminated (effect of I mode).

3- As K_c increses, the response becomes fater (effect of P and I modes) and more oscillatory to set point changes [ovesrshoot and decay ratio increase (effet of I mode)].

Large value of K_c create a very sensitive response and may lead to instability.

4- As τ_I decreases, for constant K_c, the reponse become faster but more oscillatory with higher overshoot and decay ratio (effect of I mode).

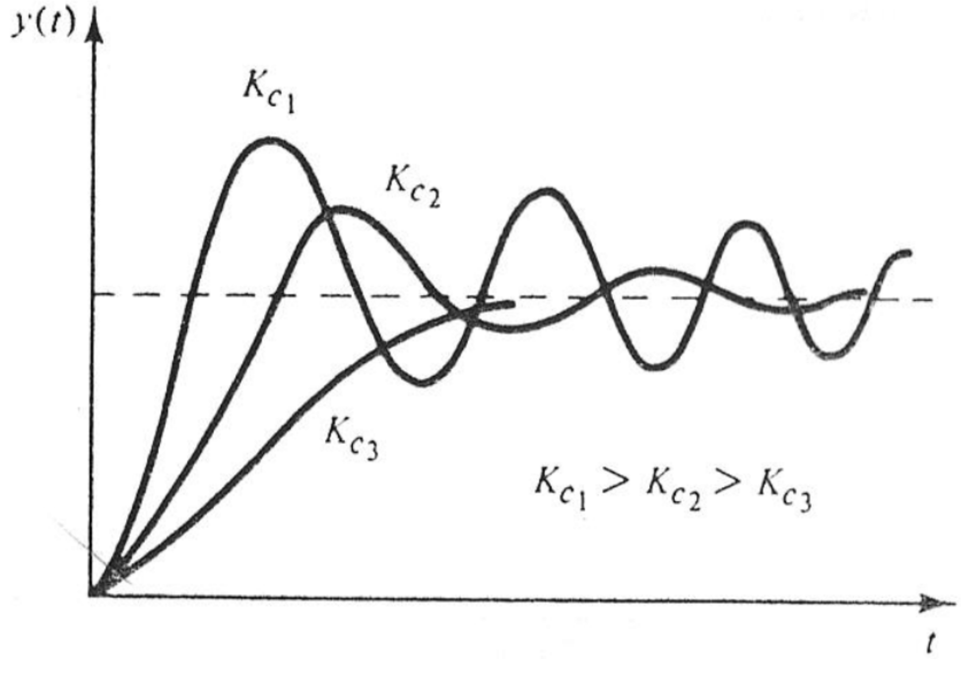
2- Effect of PID control

To increase the speed of the closed loop response, increase the value of the controller gain K_c. But increasing enough K_c in order to have acceptable speed, the response become more oscillatory and may lead to unstability.

The introduction of the derivative mode brings a stability effect to the system. Thus to achive

1- Acceptable response speed by selecting an appropriate value for the gain K_c.

2- While maintaining moderate overshoot and decay ratios.



Example: Regular loop with the following elements

$G_P(s) = \frac{3}{10s+1}$ (process)

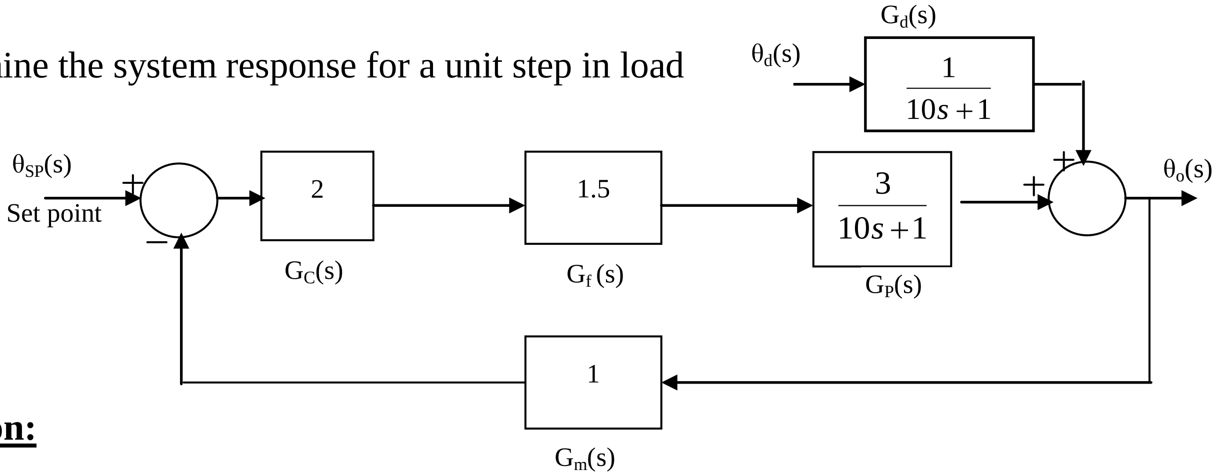
$G_d(s) = \frac{1}{10s+1}$ (Load)

$G_m(s) = 1$ (measuring device) if not given take 1

$G_C(s) = 2$ (controller)

$G_f(s) = 1.5$ (valve)

Determine the system response for a unit step in load



Solution:

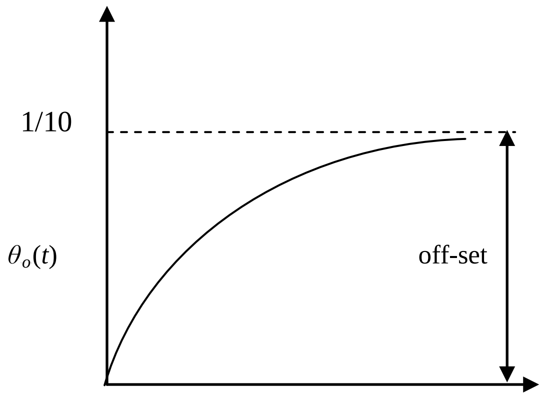
Regulator loop: $\frac{\theta_o(s)}{\theta_d(s)} = \frac{G_d(s)}{1+G_m(s)G(s)}$

$G(s) = G_C(s)\, G_f(s)\, G_P(s) = 2\times1.5\times\frac{3}{10s+1} = \frac{9}{10s+1}$

$$\frac{\theta_o(s)}{\theta_d(s)} = \frac{\frac{1}{10s+1}}{1+\frac{9}{10s+1}} = \frac{1}{10s+10}$$

$\theta_d(s) = \frac{1}{s}$

$$\theta_o(s) = \frac{\frac{1}{10}}{s(s+1)}$$



$\theta_o(t) = 0.1(1-e^{-t})$

At $t = 0$, $\theta_o(t) = 0$

At $t = \infty$, $\theta_o(\infty) = 0.1$

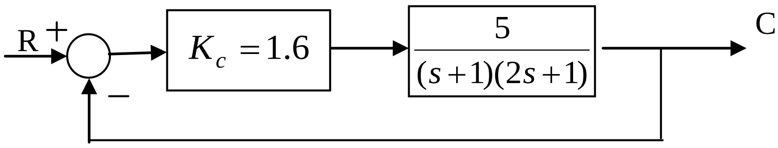
Or

$$\theta_o(\infty) = \lim_{s\to0} s\,\theta_o(s) = \lim_{s\to0} s\frac{\frac{1}{10}}{s(s+1)} = \frac{1}{10}$$

Offset= New s.s value-Ultimate value= 0 - 0.1= -0.1

Example: the set point of the control system shown in the figure is gives a step change of a 0.1 unit. Determine

1- The maximum value of C.
2- The offset.
3- The period of oscillation.



$$\frac{C(s)}{R(s)} = \frac{G_c G_p}{1 + G_c G_p}$$

$$\frac{C(s)}{R(s)} = \frac{1.6 \times \frac{5}{(s+1)(2s+1)}}{1 + 1.6 \times \frac{5}{(s+1)(2s+1)}} = \frac{8}{2s^2 + 3s + 1 + 8}$$

$$\frac{C(s)}{R(s)} = \frac{8}{2s^2 + 3s + 9} = \frac{\frac{8}{9}}{\frac{2}{9}s^2 + \frac{1}{3}s + 1} = \frac{0.8889}{0.222s^2 + 0.333s + 1}$$

$\tau^2 = 0.222 \Rightarrow \tau = 0.471$

$2\psi\tau = 0.3333 \Rightarrow \psi = 0.3538$ (Underdamped)

Ultimate Value=A*K=0.1*0.8889=0.08889

$$Overshoot = \exp\left(\frac{-\psi\pi}{\sqrt{1-\psi^2}}\right) = \exp\left(\frac{-3.1418 \times 0.3538}{\sqrt{1-(0.3538)^2}}\right) = 0.3047$$

1) The maximum value =Ultimate value*(1+*Overshoot*)

= 0.08889*(1.3047)=0.1160

To find the time required to reach maximum value apply K, A , C_{max} , ψ and τ in the equation.

$$Y(t) = kA[1 - e^{(-\psi/\tau)t}(\cos wt + \frac{\psi}{\sqrt{1-\psi^2}}\sin wt]$$

$$w = \frac{\sqrt{1-\psi^2}}{\tau}$$

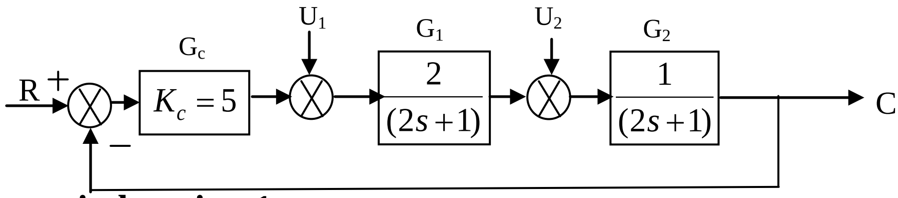
2) The offset=New set point- Ultimate value

=0.1-0.088889=0.01111

3) Period of oscillation $= \frac{2\pi\tau}{\sqrt{1-\psi^2}} = \frac{2\pi \times 0.471}{\sqrt{1-(0.3538)^2}} = 3.1640$

Example: Consider the figure below, a unit step change in load enters at either location 1 or location 2.

What is the offset when the load enters at location 1 and when it enters at location 2



a-when the load enters in location 1

$$U_1(s) = \frac{1}{s} \quad , \quad U_2(s) = 0$$

$$C(s) = \frac{G_1G_2}{1+G_cG_1G_2}U_1(s)$$

$$C(s) = \frac{\frac{2}{2s+1}\cdot\frac{1}{2s+1}}{1+\frac{2}{2s+1}\cdot\frac{1}{2s+1}\times 5}U_1(s) = \frac{2}{4s^2+4s+1+10}U_1(s)$$

$$= \frac{2}{4s^2+4s+11}U_1(s) = \frac{2/11}{\frac{4}{11}s^2+\frac{4}{11}s+1}U_1(s)$$

$$K = \frac{2}{11} = 0.1818$$

$$\tau = \sqrt{\frac{4}{11}} = 0.6030$$

$$2\psi\tau = \frac{4}{11} \Rightarrow \psi = \frac{4}{11}\times\frac{1}{2\tau} = 0.3015$$

Ultimate value=A.K=1*0.1818=0.1818

Offset=0 - 0.1818= -0.1818

b-when the load enters in location 2

$$C(s) = \frac{G_2}{1+G_cG_1G_2}U_2(s)$$

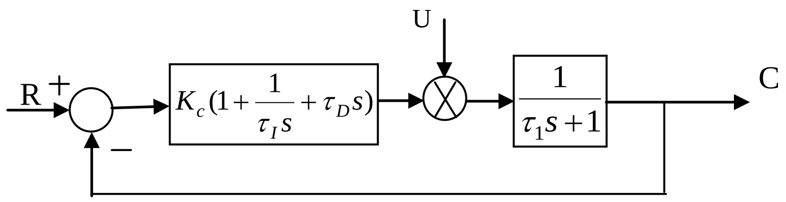
$$C(s) = \frac{\frac{1}{2s+1}}{1+\frac{2}{2s+1}\cdot\frac{1}{2s+1}\times 5}U_2(s) = \frac{2s+1}{4s^2+4s+1+10}U_2(s)$$

$$= \frac{2s+1}{4s^2+4s+11}U_2(s) = \frac{2s+1}{\frac{4}{11}s^2+\frac{4}{11}s+11}$$

$$C(\infty) = \lim_{s\to 0}\frac{2s+1}{\frac{4}{11}s^2+\frac{4}{11}s\ \ +11} = \frac{1}{11} = 0.091$$

Offset= 0 - 0.0.091= -0.091

Example: For the figure



For $\tau_D = \tau_I = 1 \quad and\, \tau_I = 2$

a- Calculate ψ when K_c=0.5 and K_c=2

b- Determine the effect for a unit-step change in load if K_c=2

$$\frac{C(s)}{R(s)} = \frac{G_c G_p}{1+G_c G_p}$$

$$\frac{C(s)}{R(s)} = \frac{K_c(1+\frac{1}{\tau_I s}+\tau_D s)\frac{1}{\tau_1 s+1}}{1+K_c(1+\frac{1}{\tau_I s}+\tau_D s)\frac{1}{\tau_1 s+1}} = \frac{K_c(1+\frac{1}{s}+s)\frac{1}{2s+1}}{1+K_c(1+\frac{1}{s}+s)\frac{1}{2s+1}}$$

$$= \frac{K_c(\frac{s+1+s^2}{s})\frac{1}{2s+1}}{1+K_c(\frac{s+1+s^2}{s})\frac{1}{2s+1}} = \frac{K_c(s+1+s^2)}{2s^2+s+K_c(s+1+s^2)} = \frac{K_c(s+1+s^2)}{(2+K_c)s^2+(1+K_c)s+K_c}$$

$$= \frac{(s+1+s^2)}{\frac{(2+K_c)}{K_c}s^2+\frac{(1+K_c)}{K_c}s+1}$$

a-1) K_c=0.5

$$\tau = \sqrt{\frac{2+K_c}{K_c}} = \sqrt{\frac{2+0.5}{0.5}} = 2.2361$$

$$2\psi\tau = \frac{(1+K_c)}{K_c} = \frac{1+0.5}{0.5} = 3 \Rightarrow \psi = \frac{3}{2\tau} = \frac{3}{2\times 2.2361} = 0.6708$$

a-2) K_c=2

$$\tau = \sqrt{\frac{2+2}{2}} = 1.4142$$

$$2\psi\tau = \frac{(1+K_c)}{K_c} = \frac{1+2}{2} = 1.5 \Rightarrow \psi = \frac{1.5}{2\tau} = \frac{1.5}{2\times 1.41421} = 0.5303$$

B) $C(s) = \frac{G_p}{1+G_p G_c} = U(s)$

$$C(s) = \frac{\frac{1}{\tau_1 s+1}}{1+K_c(1+\frac{1}{\tau_I s}+\tau_D s)\frac{1}{\tau_1 s+1}}U(s) = \frac{\frac{1}{2s+1}}{1+K_c(1+\frac{1}{s}+s)\frac{1}{2s+1}}U(s)$$

$$= \frac{s}{2s^2 + s + K_c(s+1+s^2)} U(s) = \frac{s}{2s^2 + s + 2(s+1+s^2)} \times \frac{1}{s}$$

$$= \frac{1}{4s^2 + 3s + 1}$$

$$\tau = 2$$

$$2\psi\tau = 3 \Rightarrow \psi = \frac{3}{4} = 0.75$$

$$C(\infty) = \lim_{s\to 0} s \frac{1}{4s^2 + 3s + 1} = 0$$

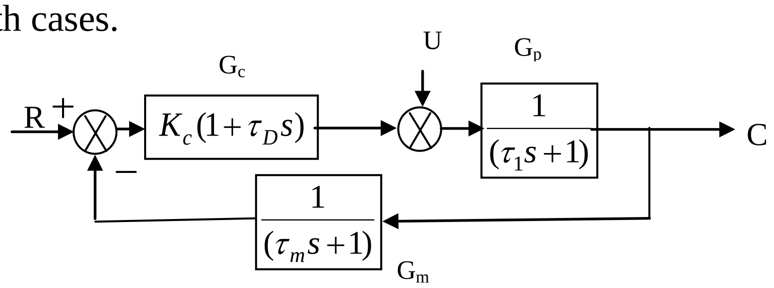
Offset=0-0=0

Example A PD controller is used in a control system having a first order process as shown. For Servo problem

a-find expression for ψ and τ for the closed loop response.

b-if τ_1=1 , τ_m=10 sec . Find K_c so that ψ=0.7 for two cases (1) τ_D=0 ,(2) τ_D=3 sec.

c- Calculate the offset in both cases.



For the closed loop T.F.

$$C = \frac{G_c G_p}{1 + G_c G_p G_m} R(s) = \frac{G_p}{1 + G_c G_p G_m} U(s)$$

$$C = \frac{K_c(1+\tau_D s) \cdot \frac{1}{\tau_1 s + 1}}{1 + K_c(1+\tau_D s) \cdot \frac{1}{\tau_1 s + 1} \cdot \frac{1}{\tau_m s + 1}} R(s)$$

$$C = \frac{K_c(1+\tau_D s)}{\tau_1 s + 1 + K_c(1+\tau_D s) \cdot \frac{1}{\tau_m s + 1}} R(s)$$

$$C = \frac{K_c(1+\tau_D s)}{\frac{\tau_1\tau_m s^2 + (\tau_1 + \tau_m)s + 1 + K_c + K_c\tau_D s}{\tau_m s + 1}} R(s)$$

$$C = \frac{K_c(1+\tau_D s)(\tau_m s + 1)}{\tau_1\tau_m s^2 + (\tau_1 + \tau_m + K_c\tau_D)s + (1+K_c)} R(s)$$

$$C = \frac{\frac{K_c(1+\tau_D s)(\tau_m s + 1)}{(1+K_c)}}{\frac{\tau_1\tau_m s^2}{(1+K_c)} + \frac{(\tau_1 + \tau_m + K_c\tau_D)}{(1+K_c)} s + 1} R(s)$$

$$\boxed{\tau = \sqrt{\frac{\tau_1 \tau_m}{1 + K_c}}}$$

$$2\psi\tau = \frac{\tau_1 + \tau_m + K_c \tau_D}{1 + K_c}$$

$$\psi = \frac{\tau_1 + \tau_m + K_c \tau_D}{2(1 + K_c)} \sqrt{\frac{1 + K_c}{\tau_1 \tau_m}}$$

$$\therefore \psi = \frac{\tau_1 + \tau_m + K_c \tau_D}{2\sqrt{(1 + K_c)\tau_1 \tau_m}}$$

b) $\therefore \psi = \dfrac{\tau_1 + \tau_m + K_c \tau_D}{2\sqrt{(1 + K_c)}\sqrt{\tau_1 \tau_m}}$ for ψ=0.7

1) τ_D=0

$$\therefore 0.7 = \frac{60 + 10 + 0}{2\sqrt{(1 + K_c)}\sqrt{60 \times 10}} = \frac{35}{\sqrt{600 + 600K_c}}$$

$$\sqrt{600 + 600K_c} = 50$$

$$600 + 600K_c = 2500$$

$$K_c = 3.166$$

2) τ_D=3 sec

$$\therefore 0.7 = \frac{60 + 10 + 3K_c}{2\sqrt{(1 + K_c)}\sqrt{600}} = \frac{70 + 3K_c}{2\sqrt{(1 + K_c)}\sqrt{600}}$$

$$70 + 3K_c = 34.292\sqrt{(1 + K_c)}$$

$$2.04(1 + 0.042K_c) = \sqrt{(1 + K_c)}$$

$$4.1616 + 0.355K_c + 0.0075K_c^2 = (1 + K_c)$$

$$0.0075K_c^2 - 0.0645K_c + 3.1616 = 0$$

$$\therefore K_c = 80.73 \quad or \quad K_c = 5.266$$

(c) The offset

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sf(s)$$

$$R = \frac{1}{s}$$

Ultimate value= $\lim_{s \to 0} s \dfrac{K_c(1 + \tau_D s)(\tau_m s + 1) / (1 + K_c)}{\dfrac{\tau_1 \tau_m s^2}{(1 + K_c)} + \dfrac{(\tau_1 + \tau_m + K_c \tau_D)}{(1 + K_c)} s + 1} \times \dfrac{1}{s} = \dfrac{K_c}{1 + K_c} = \dfrac{3.166}{4.166} = 0.76$

Offset= 1- 0.76= 0.24

Overall transfer function of a closed- loop control system

The transfer function of a block diagram is defined as the output divided by its input when represented in the Laplace domain with zero initial conditions. The transfer function G(s) of the block diagram shown in Fig. (1).

$$\frac{Y(s)}{X(s)} = G(s)$$

Here the path of the signals X(s) and Y(s) is a forward path.

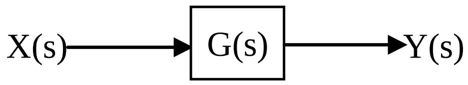


Fig. (1) Transfer function of a block diagram

Consider the block diagram of cascaded elements shown in Fig. (2a). Form the definition of a transfer function we have:

$$\frac{X_2(s)}{X_1(s)} = G_1(s)$$

$$\frac{X_3(s)}{X_2(s)} = G_2(s)$$

$$\frac{Y(s)}{X_3(s)} = G_3(s)$$

And substitution yields

$$Y(s) = G_3(s)X_3(s) = G_3(s)[G_2(s)X_2(s)] = G_3(s)G_2(s)G_1(s)X_1(s)$$

Which can be written as

$$\frac{Y(s)}{X_1(s)} = G_3(s)G_2(s)G_1(s) = G(s)$$

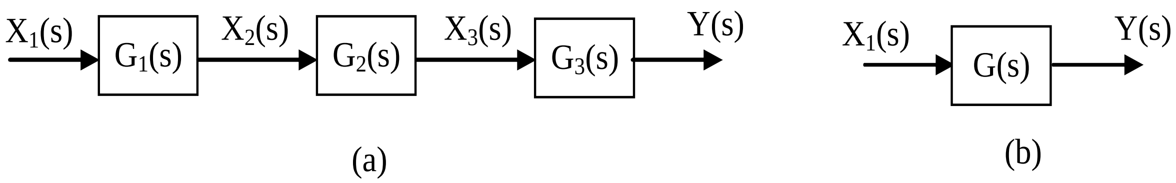


Fig. (2) Cascaded elements

The overall transfer function then is simply the product of individual transfer functions.

For applications where it is required to generate a signal which is the sum of two signals we define a summer or summing junction as shown in Fig. (3a). If the difference is required, then we define a subtractor as shown in Fig. (3b). Subtractors are often called error detecting devices since the output signal is the difference between two signals of which one is usually a reference signal.

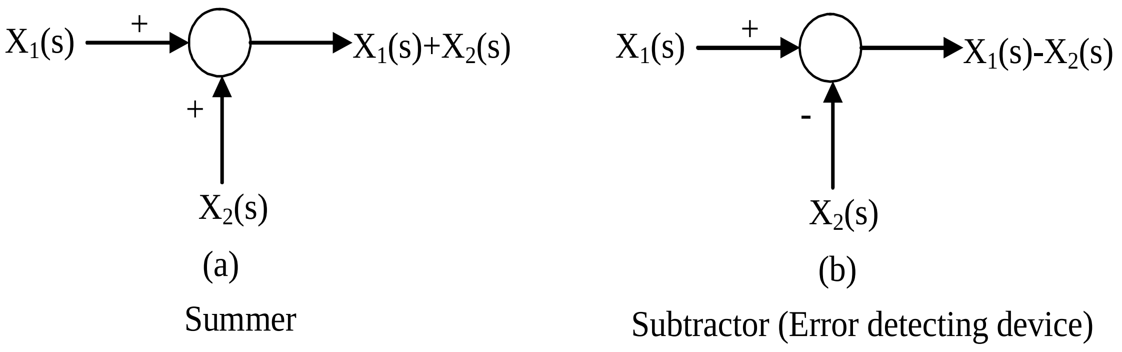


Fig. (3) Addition or subtraction of signals

The combination of block diagrams in parallel is shown in Fig. (4a). Form the definition of the transfer function we have

$$Y_1(s) = G_1(s)X(s)$$
$$Y_2(s) = G_2(s)X(s)$$
$$Y_3(s) = G_3(s)X(s)$$

And the summer adds these signals,

$$Y(s) = Y_1(s) + Y_2(s) + Y_3(s)$$

or

$$Y(s) = [G_1(s) + G_2(s) + G_3(s)]X(s)$$

The overall transfer function shown in Fig.(4b) is

$$\frac{Y(s)}{X(s)} = G(s)$$

where

$$G(s) = G_1(s) + G_2(s) + G_3(s)$$

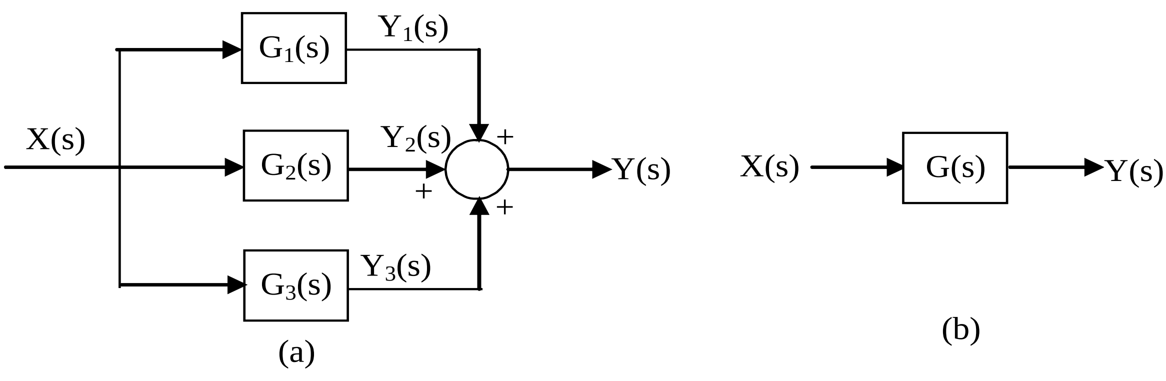


Fig. (4) Parallel combination of elements

In summary, we observe that for cascaded elements the overall transfer function is equal to the product of the transfer function of each element, whereas the overall transfer function for parallel elements is equal to the sum of the individual transfer function.

Example: Derive the overall transfer function for the control system shown in Fig. (5).

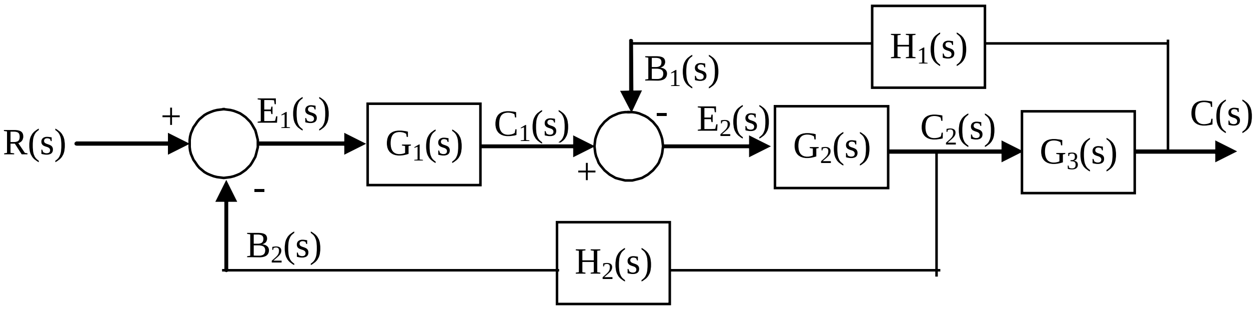


Fig.(5) Block diagram of a system with two feedback paths

Solution

$$E_1(s) = R(s) - B_2(s)$$
$$E_2(s) = C_1(s) - B_1(s)$$
$$C_1(s) = G_1(s)E_1(s)$$
$$C_2(s) = G_2(s)E_2(s)$$
$$C(s) = G_3(s)C_2(s)$$
$$B_1(s) = H_1(s)C(s)$$
$$B_2(s) = H_2(s)C_2(s)$$

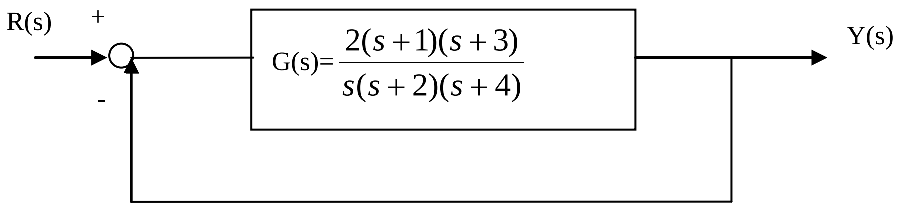
Substituting of the sub-transfer functions

$$C(s) = G_3(s)C_2(s)$$
$$C(s) = G_3(s)G_2(s)E_2(s)$$
$$C(s) = G_3(s)G_2(s)[C_1(s) - B_1(s)]$$
$$C(s) = G_3(s)G_2(s)[G_1(s)E_1(s) - H_1(s)C(s)]$$
$$C(s) = G_3(s)G_2(s)[G_1(s)(R(s) - B_2(s)) - H_1(s)C(s)]$$
$$C(s) = G_3(s)G_2(s)[G_1(s)R(s) - G_1(s)H_2(s)C_2(s) - H_1(s)C(s)]$$
$$C(s) = G_3(s)G_2(s)[G_1(s)R(s) - G_1(s)H_2(s)\frac{C(s)}{G_3(s)} - H_1(s)C(s)]$$
$$C(s) = G_3(s)G_2(s)G_1(s)R(s) - G_3(s)G_2(s)G_1(s)H_2(s)\frac{C(s)}{G_3(s)} - G_3(s)G_2(s)H_1(s)C(s)$$
$$[1 + G_2(s)G_1(s)H_2(s) + G_3(s)G_2(s)H_1(s)]C(s) = G_3(s)G_2(s)G_1(s)R(s)$$

Finally, the overall transfer function

$$\frac{C(s)}{R(s)} = \frac{G_1(s)G_2(s)G_3(s)}{1 + G_1(s)G_2(s)H_2(s) + G_2(s)G_3(s)H_1(s)}$$

Example: A single-loop control system is shown in figure below. Determine closed-loop transfer function $\frac{Y(s)}{R(s)}$



Solution

Transfer function $\frac{Y(s)}{R(s)} = \frac{G}{1+GH}$

$$\frac{Y(s)}{R(s)} = \frac{\frac{2(s+1)(s+3)}{s(s+2)(s+4)}}{1+\frac{2(s+1)(s+3)}{s(s+2)(s+4)}*1} = \frac{\frac{2(s+1)(s+3)}{s(s+3)(s+4)}}{\frac{s(s+2)(s+4)+2(s+1)(s+3)}{s(s+2)(s+4)}}$$

$$\frac{2(s+1)(s+3)}{s(s+2)(s+4)+2(s+1)(s+3)} = \frac{2s^2+6s+2s+6}{s^3+4s^2+2s^2+8s+2s^2+6s+2s+6}$$

$$\therefore \frac{Y(s)}{R(s)} = \frac{2s^2+8s+6}{s^3+8s^2+16s+6}$$

Block Diagram Reduction

When the block diagram representation gets complicated, it is advisable to reduce the diagram to a simpler and more manageable form prior to obtaining the overall transfer function. We shall consider only a few rules for block diagram reduction. We have already two rules, viz. Cascading and parallel connection.

Consider the problem of moving the starting point of a signal shown in Fig. (6a) from behind to the front of $G(s)$. since $B(s)=R(s)$ and $R(s)=C(s)/G(s)$, then $B(s)=C(s)/G(s)$. therefore if the takeoff point is in front of $G(s)$, then the signal must go through a transfer function $1/G(s)$ to yield $B(s)$ as shown in Fig. (7b).

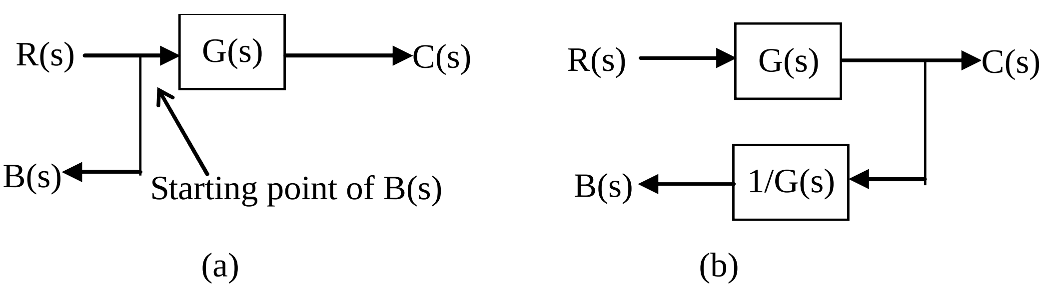


Fig.6 Moving the starting point of a signal

Consider the problem of moving the summing point of Fig. (7a). Since

$$E(s) = [M(s) + C(s)]G(s) = M(s)G(s) + C(s)G(s)$$

$$E(s) = M_1(s) + C_1(s)$$

where

$$M_1(s) = M(s)G(s); \qquad C_1(s) = C(s)G(s)$$

The generation of the signals $M_1(s)$ and $C_1(s)$ and adding them to yield $E(s)$ is shown in Fig. (7b). A table of the most common reduction rules is given in Table 1.

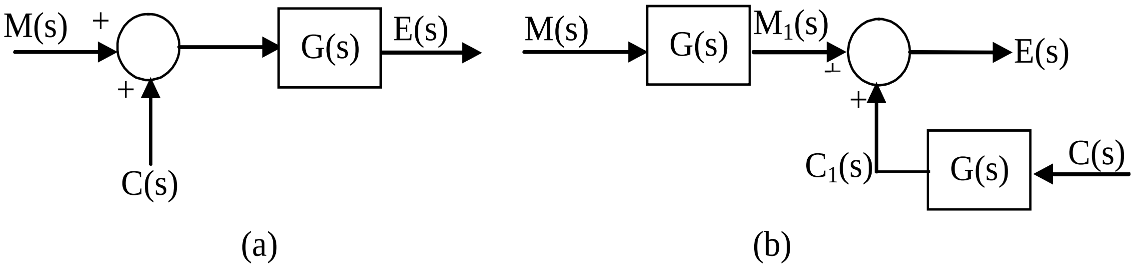
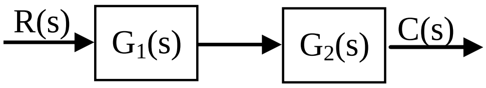
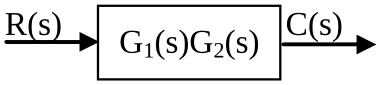
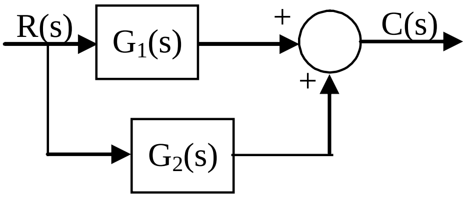
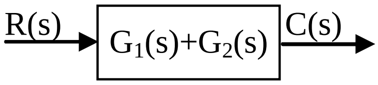
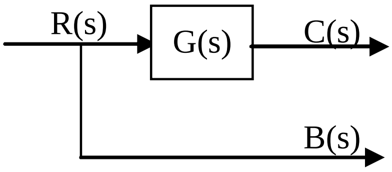
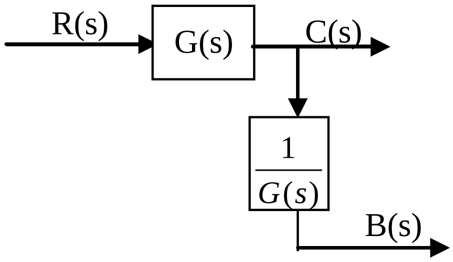
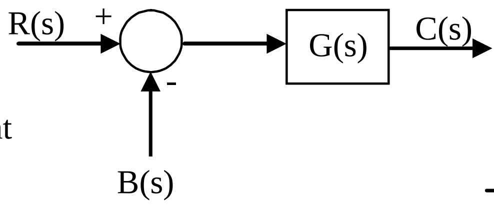
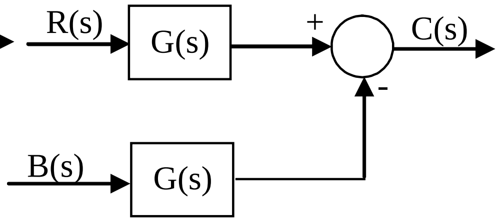
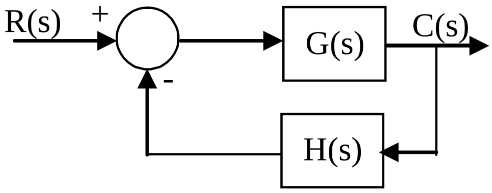
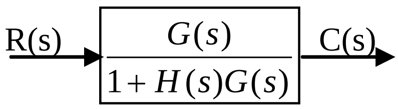


Fig.(7) Moving a summing junction

Table 1 Some rules for block diagram reduction

Rule	Original system	Reduced system
Cascaded elements		
Addition or subtraction		
Moving a starting point		
Moving a summing point		
Closed loop system		

Consider the transfer function of the system shown in Fig. (8a). The final transfer function is shown in Fig. (8d).

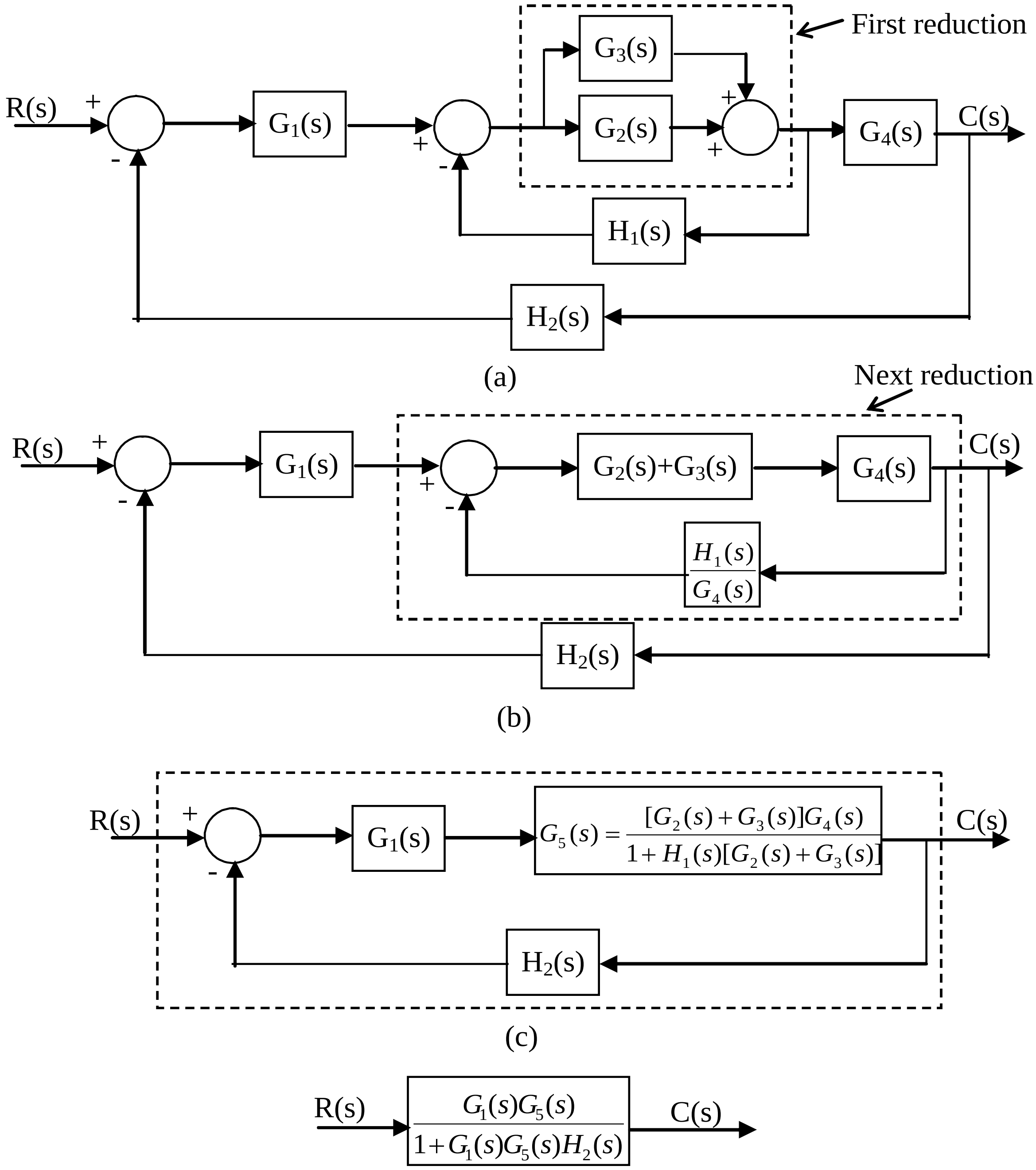
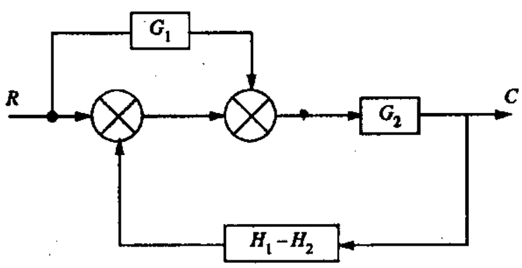
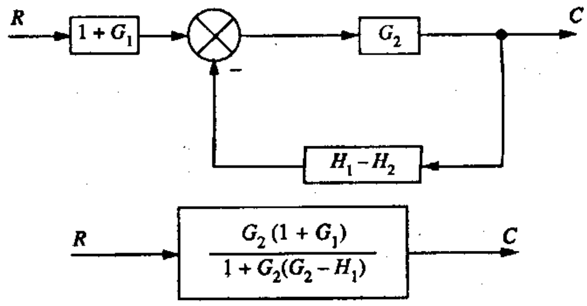


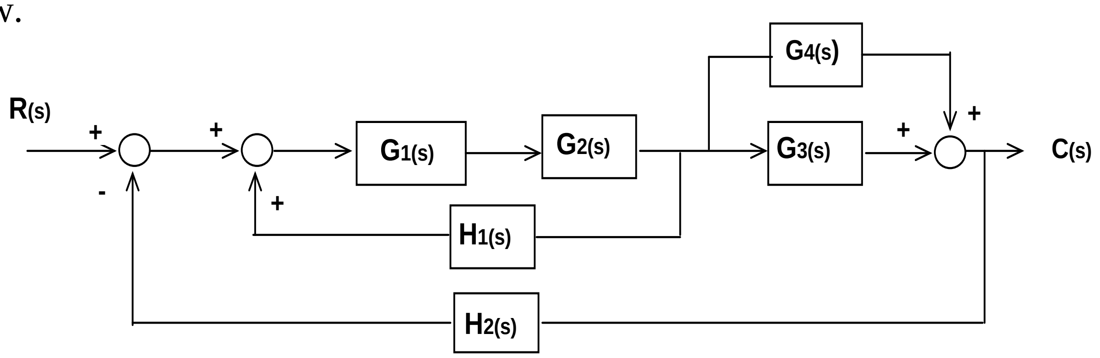
Fig.8 Obtaining transfer function by block diagram reduction

Example: Obtain the transfer function C/R of the block diagram shown in Figure below.

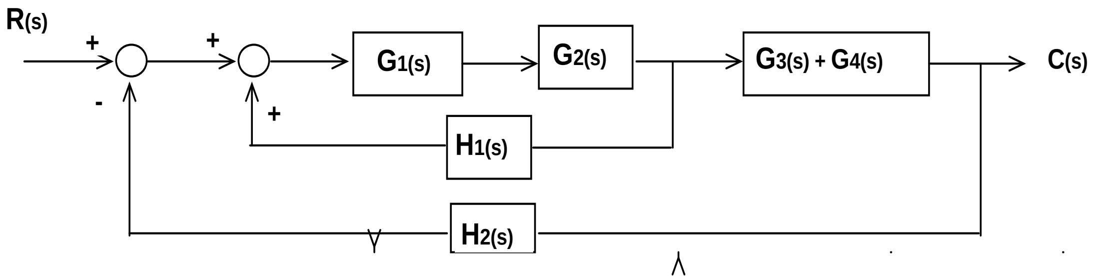


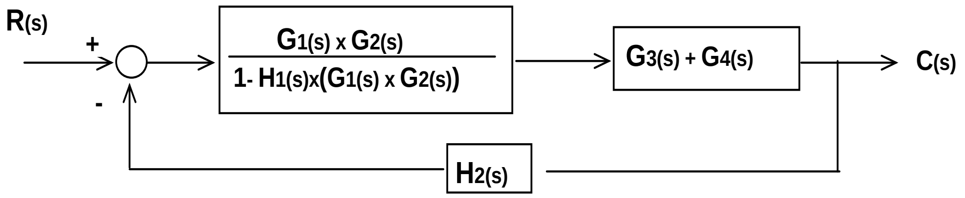


Example: Obtain the transfer function C/R of the block diagram shown in Figure below.



Solution:



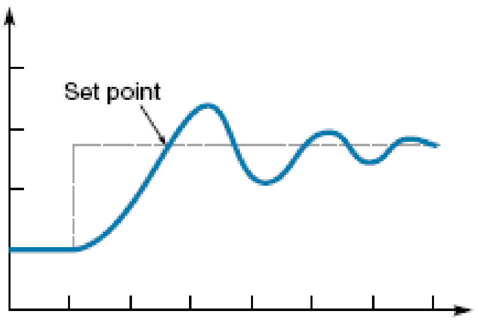


$$\frac{C(s)}{R(s)}=\frac{\dfrac{G_1(s).G_2(s)\left(G_3(s)+G_4(s)\right)}{1-H_1(s).G_1(s).G_2(s)}}{1+H_2(s)\left[\dfrac{G_1(s).G_2(s).\left(G_3(s)+G_4(s)\right)}{1-H_1(s).G_1(s).G_2(s)}\right]}$$

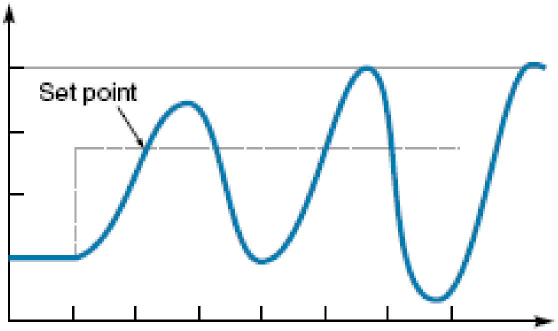
$$=\frac{\dfrac{G_1(s).G_2(s)\left(G_3(s)+G_4(s)\right)}{1-H_1(s).G_1(s).G_2(s)}}{\dfrac{1-H_1(s).G_1(s).G_2(s)}{1-H_1(s)\cdot G_1(s)\cdot G_2(s)}+H(s)\dfrac{G_1(s).G_2(s)\left(G_3(s)+G_4(s)\right)}{1-H_1(s).G_1(s).G_2(s)}}$$

$$\frac{C(s)}{R(s)}=\frac{\left(G_1(s)\cdot G_2(s)\right)\times\left(G_1(s)+G_4(s)\right)}{\left(1-H_1(s)\cdot G_1(s)\cdot G_2(s)\right)+\left(H_2(s)\cdot G_1(s)\cdot G_2(s)\right)\times\left(G_3(s)+G_4(s)\right)}$$

A stable system is one where the controlled variable will always settle near the set point. An unstable system is one where, under some conditions, the controlled variable drifts away from the set point or breaks into oscillations that get larger and larger until the system saturates on each side.



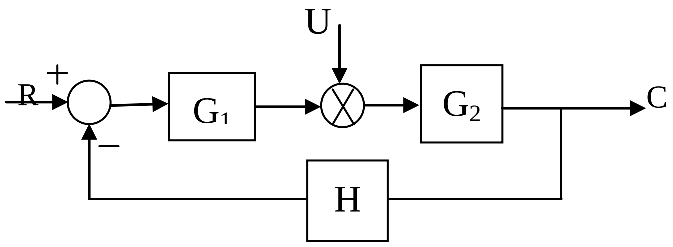
Stable system



unstable system

<u>Methods of Stability Test</u>

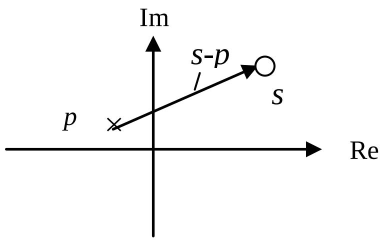
1-Determination the roots of equation



$$C = \frac{G_1G_2}{1+G_1G_2H}R(s) + \frac{G_2}{1+G_1G_2H}U(s)$$

$$1 + G_1G_2H = 0 \text{ (Characterstic Equation)}$$

$$(s-r_1)(s-r_2)(s-r_3)\ldots\ldots\ldots = 0$$



A linear control system is ***unstable*** if any roots of its characterstic equation are to the right of imaginary axis.

If this equation has some roots with positive real parts, then the system is unstable, or some roots equal to zero, the system is marginally stable (oscillatory), therefore it is unstable.

Then for stability the roots of characteristic equation must have negative real parts.

Example: if

$$G_1 = 10\frac{0.5s+1}{s} \quad \text{PI control}$$

$$G_2 = \frac{1}{2s+1} \qquad \text{Stirred tank}$$

$$H = 1 \qquad \text{Mesuring element without lag}$$

$$1 + G = 1 + G_1 G_2 H = 0$$

$$1 + \frac{10(0.5s+1)}{s(2s+1)} = 0$$

$$s(2s+1) + 5s + 10 = 0$$

$$2s^2 + 6s + 10 = 0$$

$$s^2 + 3s + 5 = 0$$

$$s = \frac{-3}{2} \mp \frac{\sqrt{9-20}}{2}$$

$$\therefore s_1 = \frac{-3}{2} + j\frac{\sqrt{11}}{2} \quad \text{and} \quad s_2 = \frac{-3}{2} - j\frac{\sqrt{11}}{2}$$

Since the real part in s_1 and s_2 in -ve $(-\frac{3}{2}) \therefore$ The system is stable

2-Routh's Method

a-Write the characterstic eqn. on the form of a polynomial shape:

$$a_o s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots\ldots a_n = 0 \qquad (*)$$

Where a_o is positive

It is necessary that a_o, a_1, a_2,…. a_{n-1}, a_n be positive. If any coeff. is negative, the system is ***unstable***.

If all of the coeff. are positive, the system may be stable or unstable. Then apply the next step.

b. Routh array:

Arrange the coeff. of eqn. (*) into the first two rows of the Routh array shown below.

Row				
1	a_o	a_2	a_4	a_6
2	a_1	a_3	a_5	a_7
3	A_1	A_2	A_3	
4	B_1	B_2	B_3	
n+1	C_1	C_2	C_3	

$$A_1 = \frac{a_1 a_2 - a_o a_3}{a_1} \quad , \quad A_2 = \frac{a_1 a_4 - a_o a_5}{a_1} \quad , \quad A_3 = \frac{a_1 a_6 - a_o a_7}{a_1}$$

$$B_1 = \frac{A_1 a_3 - a_1 A_2}{A_1} \quad , \quad B_2 = \frac{A_1 a_5 - a_1 A_3}{A_1}$$

$$C_1 = \frac{B_1 A_2 - A_1 B_2}{B_1} \quad , \quad C_2 = \frac{B_1 A_3 - A_1 B_3}{B_1}$$

Examine the elements of the first column of the array a_o, a_1, A_1, B_1,C_1………W_1

a) If any of these elements is negative, we have at least one root on the right of the imaginary axis and the system is unstable.
b) The number of sign changes in the elemnts of the first column is equal to the number of root to the right of the imaginary axis.

$\therefore$ The system is ***stable*** if all the elements in the first column of the array are positive

<u>Example</u>: Given the characteristic eqn.

$s^4 + 3s^3 + 5s^2 + 4s + 2 = 0$

<u>Solution:</u>

Row			
1	1	5	2
2	3	4	0
3	11/3	2	0
4	2.36	0	
5	2		

$$A_1 = \frac{3 \times 5 - 4 \times 11}{3} = \frac{11}{3}$$

$$A_2 = \frac{3 \times 2 - 0}{3} = 2$$

$$B_1 = \frac{11/3 \times 4 - 6}{11/3} = 2.36$$

$$C_1 = \frac{2.36 \times 2}{2.36} = 2$$

$\therefore$ The system is **stable**

<u>Example</u>: Apply the Routh's stability criterion to the equation:

$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$

<u>Solution:</u>

s^4	1	3	5
s^3	2	4	
s^2	1	5	
s^1	-6	0	
s^0	5		

The system is unstable.

Example: A system has a characteristic equation $s^3+9s^2+26s+24=0$. Using the Routh criterion, show that the system is stable.

Solution

q(s) = $s^3+9s^2+26s+24$

Using the Routh-Hurwitz criterion,

$$\begin{array}{c|cc} s^3 & 1 & 26 \\ s^2 & 9 & 24 \\ s^1 & 26 & 0 \\ s^0 & 24 & 0 \end{array}$$

No sign change in 1^{st} column then the system is stable.

Example: Consider the feedback control system with the characteristic equation.

$$s^3+2s^2+(2+K_c)s+\frac{K_c}{\tau_I}=0$$

Solution:

The corresponding Routh array can now be formed

Row			
1	1	$2+K_c$	0
2	2	$\frac{K_c}{\tau_I}$	0
3	$\frac{2(2+K_c)-K_c/\tau_I}{2}$	0	0
4	K_c/τ_I	0	0

The elemnets of the first-column are positive except the third, which can be positive or negative depending on K_c and τ_I.

So state the stability

Put $\frac{2(2+K_c)-K_c/\tau_I}{2}>0 \quad \Rightarrow 2(2+K_c)>\frac{K_c}{\tau_I}$

If K_c and τ_I satisfy the condition, then the system is atable

Critical stability

Put the third element=0

i.e $2(2+K_c)=\frac{K_c}{\tau_I}$

For τ_I=0.1

$2(2+K_c)=10K_c \Rightarrow 4=8K_c$

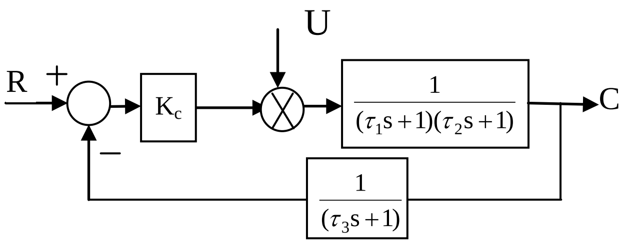
$K_c=0.5$

1) if $K_c < 0.5$, the system is stable (all of the elements in the 1st column is +ve)

2) if $K_c > 0.5$, the third element of the 1st column is negative. We have two sign change in the elements of the first column.

$\therefore$ we have two roots to the right of imaginary axis.

Example:



If $\tau_1 = 1, \tau_2 = \frac{1}{2}, \tau_3 = \frac{1}{3}$

Determine K_c for a stable system

Solution:

The char. Eqn.

$$1 + K_c \frac{1}{(s+1)(\frac{1}{2}s+1)(\frac{1}{3}s+1)} = 0$$

$$(s+1)(\frac{1}{2}s+1)(\frac{1}{3}s+1) + K_c = 0$$

$$(\frac{1}{2}s^2 + \frac{3}{2}s + 1)(\frac{1}{3}s+1) + K_c = 0$$

$$\frac{s^3}{6} + \frac{s^2}{2} + \frac{s}{3} + \frac{s^2}{2} + \frac{3s}{2} + 1 + K_c = 0$$

$$\frac{1}{6}s^3 + s^2 + \frac{11}{6}s + 1 + K_c = 0$$

Row		
1	1/6	11/6
2	1	$1+K_c$
3	$\frac{10-K_c}{6}$	0
4	$1+K_c$	

Since $K_c>0$

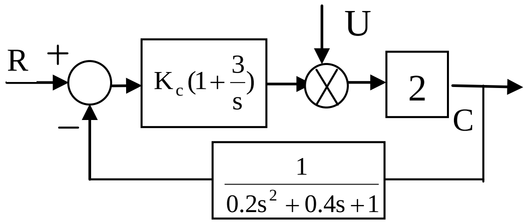
$\therefore$ The system will be stable

If $10-K_c>0$

$K_c < 10$

Therfore K_c must within the range $0<K_c< 10$

Example:



Study the stability for K_c=2

Solution:

$$1+K_c(1+\frac{3}{s})\times 2\times\frac{1}{0.2s^2+0.4s+1}=0$$

$$1+K_c(\frac{s+3}{s})\times\frac{2}{0.2s^2+0.4s+1}=0$$

$$1+(\frac{sK_c+3K_c}{s})\times\frac{2}{0.2s^2+0.4s+1}=0$$

$$0.2s^3+0.4s^2+s+2sK_c+6K_c=0$$

$$0.2s^3+0.4s^2+(1+2K_c)s+6K_c=0$$

Row		
1	0.2	$1+2K_c$
2	0.4	$6K_c$
3	A_1	0
4	B_1	0

Row	For K_c=2	
1	0.2	5
2	0.4	12
3	$\frac{2-2.4}{0.4}$	0
4	1.2	0

$$A_1=\frac{0.4(1+2K_c)-(1.2K_c)}{0.4}=\frac{0.4+0.8K_c-1.2K_c}{0.4}=\frac{0.4-0.4K_c}{0.4}$$

$$0.4-0.4K_c>0$$

The system is stable for $K_c<1$

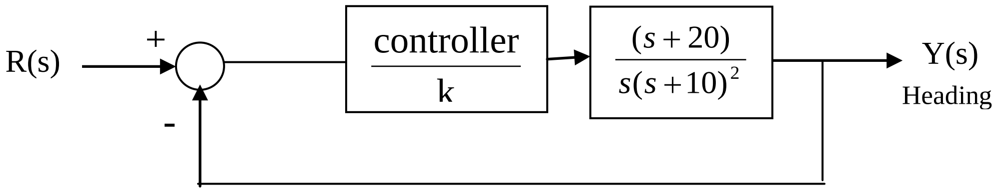
$B_1=6K_c\Rightarrow 6K_c>0$

And $K_c>0$

Therfore K_c must within the range $0<K_c< 1$

Example: Designers have developed small, fast, vertical-take off fighter aircraft that are invisible to radar. This aircraft concept uses quickly turning jet nozzles to steer the airplane. The control system for the heading or direction control is shown in figure.

Determine the maximum gain of the system for stable operation.



Solution

$$G(s) = \frac{k(s+20)}{s(s+10)^2} = \frac{ks+20k}{s(s^2+20s+100)} = \frac{ks+20k}{s^3+20s^2+100s}$$

Characteristic equation,

1+GH = 0

$$1+\frac{ks+20k}{s^3+20s^2+100s}*1=0$$

$$s^3+20s^2+100s+ks+20k=0$$

$$s^3+20s^2+(100+k)s+20k=0$$

The corresponding Routh array can now be formed

Row		
1	1	100+k
2	20	20k
3	a	0
4	b	0

$$a = \frac{20(100+k)-20k}{20} = \frac{20*100+20k-20k}{20} = 100$$

$$b = \frac{a*20k-0}{a} = 20\,k$$

The system is stable, no sign change in 1^{st} column,

b>0

20k >0

k>0

$\therefore$ Range of k is must be k>0

Frequency Response Analysis

It is how the output response (amplitude, phase shift) change with the frequency of the input sinusoidal.

The input frequency is varied, and the output characteristics are computed or represented as a function of the frequency. Frequency response analysis provides useful insights into stability and performance characteristics of the control system. Figure below shows the hypothetical experiment that is conducted.

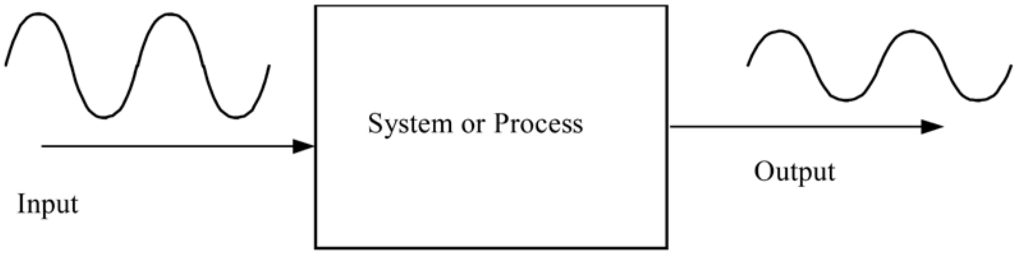


Figure: How frequency response is defined.

Response of a first-Order System to a Sinusoidal Input

Consider a simple first-order system with a transfer function

$$G(s) = \frac{\bar{y}(s)}{\bar{F}(s)} = \frac{K_p}{\tau_p s + 1} \qquad (1)$$

Let F(t) be a sinusoidal input with amplitude A and frequency ω;

$F(t) = A \sin(\omega t)$

Then

$$\bar{F}(s) = \frac{A\omega}{s^2 + \omega^2} \qquad (2)$$

Sub. $\bar{F}(s)$ from eq. (2) into eq. (1)

$$\bar{y}(s) = \frac{K_p}{\tau_p s + 1} \times \frac{A\omega}{s^2 + \omega^2} = \frac{K_p}{\tau_p s + 1} \times \frac{A\omega}{(s + j\omega)(s - j\omega)}$$

Expand into partial fraction and find

$$\bar{y}(s) = \frac{C_1}{s + 1/\tau_p} + \frac{C_2}{s + j\omega} + \frac{C_3}{s - j\omega}$$

Compute the constants C_1, C_2 and C_3 and find the inverse of laplace transform.

$$C_1 = \frac{K_p A \omega \tau_p}{\tau_p^2 \omega^2 + 1} \quad , \quad C_2 = \frac{-K_p A \omega \tau_p}{\tau_p^2 \omega^2 + 1} \quad , \quad C_3 = \frac{K_p A}{\tau_p^2 \omega^2 + 1}$$

$$\bar{y}(t) = \frac{K_p A \omega \tau_p}{\tau_p^2 \omega^2 + 1} e^{-t/\tau_p} - \frac{K_p A \omega \tau_p}{\tau_p^2 \omega^2 + 1} \cos(\omega t) + \frac{K_p A}{\tau_p^2 \omega^2 + 1} \sin(\omega t)$$

As $t \to \infty$, then $e^{-t/\tau_p} \to 0$, and the first term disappers.

Thus, after a long time, the response of a first order system to a sinusoidal input is given by:

$$\bar{y}_{ss}(t) = -\frac{K_p A \omega \tau_p}{\tau_p^2 \omega^2 + 1}\cos(\omega t) + \frac{K_p A}{\tau_p^2 \omega^2 + 1}\sin(\omega t)$$

$$\bar{y}_{ss}(t) = \frac{K_p A}{\tau_p^2 \omega^2 + 1}[-\omega \tau_p \cos(\omega t) + \sin(\omega t)] \quad (3)$$

Use the follwing trigonometric identity.

$$\boxed{\begin{aligned} &p\cos\theta + q\sin\theta = r\sin(\theta + \phi) \\ &r = \sqrt{p^2 + q^2} \qquad \phi = \tan^{-1}\frac{p}{q} \end{aligned}}$$

$$q = 1 \qquad\qquad p = -\omega\, \tau_p$$

$$r = \sqrt{(-\tau_p \omega)^2 + (1)^2} = \sqrt{\tau_p^2 \omega^2 + 1}$$

$$\phi = \tan^{-1}\frac{p}{q} = \tan^{-1}\left(\frac{-\omega\, \tau_p}{1}\right) = \tan^{-1}(-\omega\, \tau_p)$$

Then eq.(3) yield

$$\bar{y}_{ss}(t) = \frac{K_p A}{\tau_p^2 \omega^2 + 1}[(\sqrt{\tau_p^2 \omega^2 + 1})\sin(\omega t + \phi)]$$

$$\bar{y}_{ss}(t) = \frac{K_p A}{\sqrt{\tau_p^2 \omega^2 + 1}}\sin(\omega t + \phi) \quad (4)$$

$$\boxed{\phi = \tan^{-1}(-\omega \tau_p)} = \text{Phase lag} \quad (5)$$

From eq.(4) and eq. (5), we observe that:

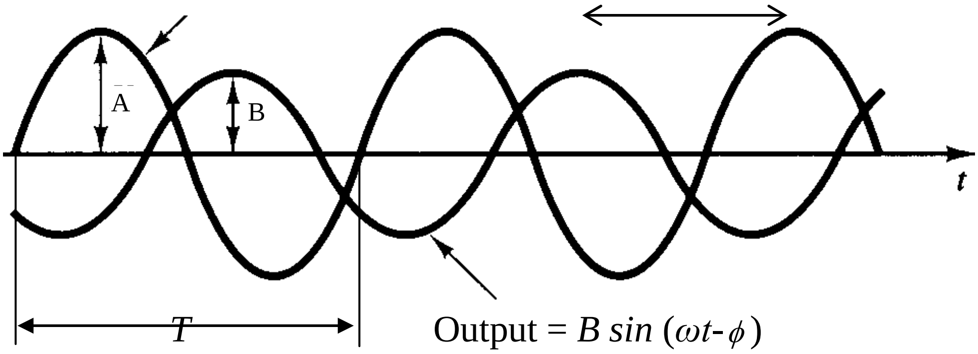
1) The ultimate response (also referred to as s.s.) of a first order system to a sin input is also a sinusoidal wave with the same frequency ω.
2) The ratio of the output amplitude to the input amplitude is called the " amplitude ratio" and is a function of the frequency:

$$AR = \text{amplitude ratio} = \frac{\dfrac{K_p A}{\sqrt{\tau_p^2 \omega^2 + 1}}}{A} = \frac{K_p}{\sqrt{\tau_p^2 \omega^2 + 1}}$$

$$\boxed{AR = \frac{K_p}{\sqrt{\tau_p^2 \omega^2 + 1}}} \quad (6)$$

3) The output wave lags behind (phase lag) the input wave by an angle $|\phi|$>, which is a function of the frequency ω (see eq.(5)).

   Input = *A sin (ωt)* ϕ



It is the most important methods for stability analysis and used for design purposes control system.
Suppose the input to the process is sinusoidal signal
Where: A is amplitude
ω is frequency (rad/sec) = $\frac{1}{T}$
T is period of one complete cycle (time)

Frequency Response of a Second Order System

For a second order system the transfer function is:

$$G(s)=\frac{K_p}{\tau^2 s^2+2\psi\tau s+1}$$

Put s=jω then

1) Amplitude Ratio

$$AR=\frac{K_p}{\sqrt{(1-\tau^2\omega^2)^2+(2\psi\tau\omega)^2}}$$

2) Phase shift

$$\phi=\tan^{-1}\left(-\frac{2\psi\tau\omega}{1-\tau^2\omega^2}\right)$$

Which is aphase lag since $\phi<0$

Frequency Response of a Pure Dead-Time Process

$$G(s)=e^{-t_d s}$$

Put s=jω

$$\therefore\quad G(j\omega)=e^{-jt_d\omega}$$

$$AR=1$$

$$\phi=-t_d\omega$$

That is aphase lag since $\phi<0$

Frequency Response of a Feedback Controllers

1- Propertional controller:

The transfer function is $G(s) = K_c$

$$\therefore \quad \boxed{AR = K_c} \quad \boxed{\phi = 0}$$

2- PI controller:

The transfer function is $G(s) = K_c(1 + \frac{1}{\tau_I s})$

$$\therefore \quad \boxed{AR = K_c\sqrt{1 + \frac{1}{(\omega\tau_I)^2}}}$$

$$\boxed{\phi = \tan^{-1}(\frac{-1}{\omega\tau_I})} \quad < 0$$

3- PD controller:

The transfer function is $G(s) = K_c(1 + \tau_D s)$

$$\boxed{AR = K_c\sqrt{1 + \tau_D^2\omega^2}}$$

$$\boxed{\phi = \tan^{-1}(\tau_D \omega)} \quad > 0$$

The positive phase shift is called phase lead and implies that the controller output lead the input.

4-PID controller:

The transfer function is $G(s) = K_c(1 + \frac{1}{\tau_I s} + \tau_D s)$

$$\boxed{AR = K_c\sqrt{(\tau_D\omega - \frac{1}{\tau_I\omega})^2 + 1}}$$

$$\boxed{\phi = \tan^{-1}(\tau_D\omega - \frac{1}{\tau_I\omega})}$$

ϕ is + or – ve depending on the values of τ_D, τ_I and ω

Bode Diagrams

The bode diagrams consist of a pair of plots showing:
1. How the logarithm of the amplitude ratio varies with frequency.
2. How the phase shift varies with frequency.

<u>First Order system:</u>

Amplitude ratio $AR = \dfrac{K_p}{\sqrt{1+\tau_p^2\omega^2}}$ (*)

Phase lag= $\phi = \tan^{-1} - \tau_p\,\omega$

$$\log\frac{AR}{K_p} = -\frac{1}{2}\log(1+\tau_p^2\,\omega^2) \qquad (**)$$

The plot can be carried by considering its asymptotic behaviour as **ω→0** and as **ω→∞** . Then

1. As ω→0, then $\tau_p\,\omega \to 0$ and from eq.(*)

$\log\dfrac{AR}{K_p} \to 0$ or $\dfrac{AR}{K_p} = 1$.This is the low-frequency asymptote. It is a horizontal line passing through the point $\dfrac{AR}{K_p} = 1$.

2. As ω→∞, then $\tau_p\,\omega \to \infty$ and from eq.(**)

$\log\dfrac{AR}{K_p} = -\log\tau_p\,\omega$. This is the high frequency asymptote.

It is a line with slope -1 passing through the point $\dfrac{AR}{K_p} = 1$ for $\tau_p\,\omega = 1$.

3. At the corner $\tau_p\omega = 1 \to \omega = \omega_c$

$$\omega_{corner} = \omega_c = \frac{1}{\tau_p}$$

The frequency ω_c is known as the corner frequnecy (and $\dfrac{AR}{K_p} = \dfrac{1}{\sqrt{2}} = 0.707$)

<u>The phase lag plot</u>

$as\ \omega \to 0 \quad , \quad \phi \to 0$

$as\ \omega \to \dfrac{1}{\tau_p} \quad , \quad \phi \to \tan^{-1}(-1) = -45^o$

$as\ \omega \to \infty \quad , \quad \phi \to \tan^{-1}(-\infty) = -90^o$

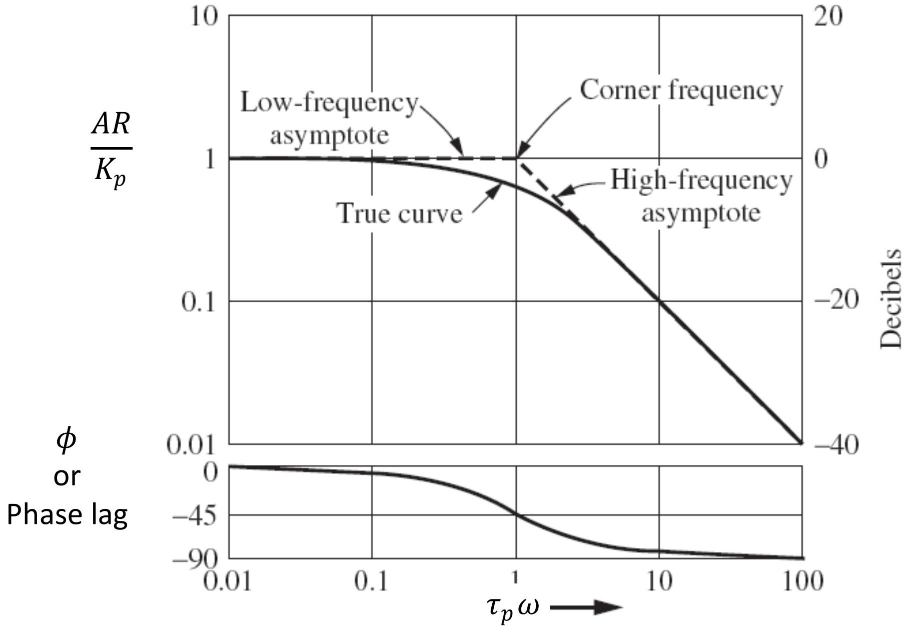


Figure: Bode diagram for first-order system.

Second –order system

$$AR = \frac{K_p}{\sqrt{(1-\tau^2\omega^2)^2 + (2\psi\tau\omega)^2}} \quad \phi = \tan^{-1}\left(\frac{-2\psi\tau\omega}{1-\tau^2\omega^2}\right)$$

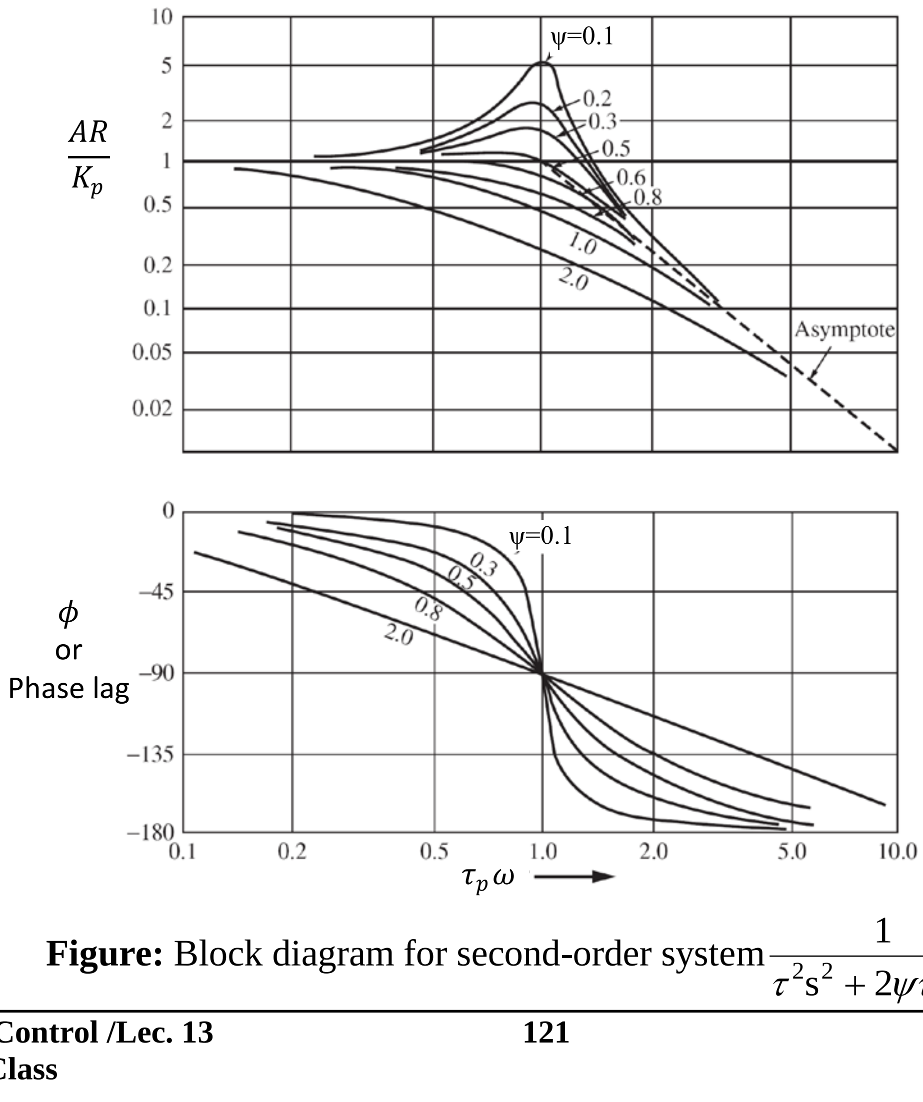


Figure: Block diagram for second-order system $\frac{1}{\tau^2 s^2 + 2\psi\tau s + 1}$

$$\log\frac{AR}{K_p} = -\frac{1}{2}\log[(1-\tau^2\omega^2)^2 + (2\psi\tau\omega)^2]$$

1) as $\omega \to 0$, then $\log\frac{AR}{K_p} = -\frac{1}{2}\log(1) = 0$

$\therefore \frac{AR}{K_p} = 1$ stright line of a slope=0 **(L.F.A)**

$$\phi = \tan^{-1}\frac{-0}{1} = 0$$

2) as $\omega \to \infty$, then $\log\frac{AR}{K_p} = -\frac{1}{2}\log(\tau\omega)^4 = -2\log(\tau\omega)$ **(H.F.A)**

It is a straight line with a slope -2 passing through the pointAR=1 and τω=1

3) $\omega = \omega_c = \frac{1}{\tau}$

Pure dead-time system

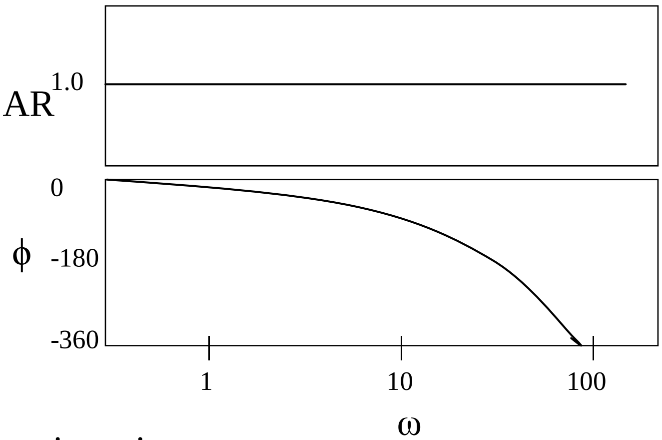
For the system

$AR = 1$

$\phi = -\tau_d\,\omega$

as $\omega \to 0$, $\phi = 0$

as $\omega \to \infty$, $\phi = \infty$



Example: Two systems in series

$$G_1(s) = \frac{1}{2s+1} \text{ and } G_2(s) = \frac{6}{5s+1}$$

The overall T.F. is

$$G(s) = \frac{1}{2s+1}\cdot\frac{6}{5s+1}$$

$$\therefore AR = \frac{1}{\sqrt{1+4\omega^2}}\cdot\frac{6}{\sqrt{1+25\omega^2}}$$

$$\log AR = \log 6 + \log(AR)_1 + \log(AR)_2$$

1- Region 1: From ω=0 to $\omega = \frac{1}{5}$, slope of the overall asymptote =0+0=0 (i.e. horizontal * going through the point AR=1)

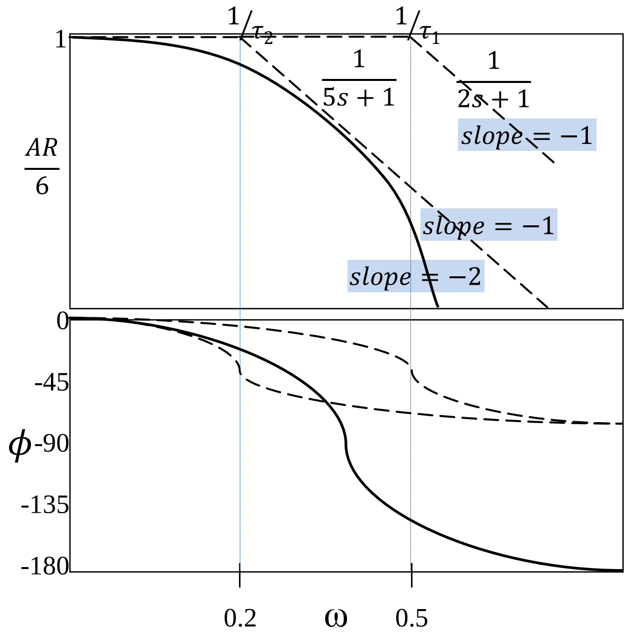
2- Region 2: From $\omega = \frac{1}{5}$ to $\omega = \frac{1}{2}$, slope of the overall asymptote =0+(-1)=-1 going through the point AR=1 , $\omega = \frac{1}{5}$

3- Region 3: From $\omega > \frac{1}{2}$, slope of the overall asyptote =(-1)+(-1)=-2

For ϕ

When as $\omega \to 0$, $\phi_1 \to 0$, $\phi_2 \to 0$, $\phi \to 0$

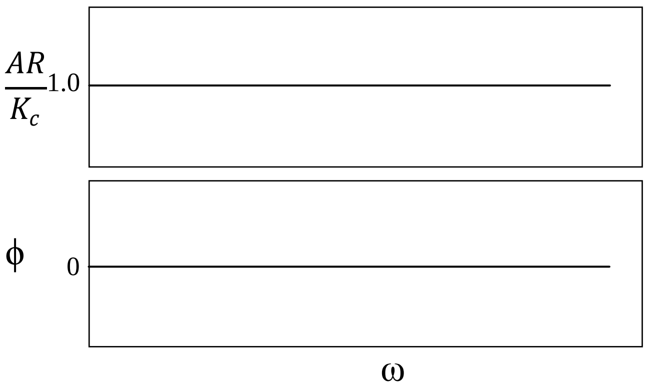
When as $\omega \to \infty$, $\phi_1 \to -90$, $\phi_2 \to -90$, $\phi \to -180$



Feedback Controller

1-Propertional controller

$AR = K_c \; \phi = 0$



2-Propertional Integral controller (PI)

$$AR = K_c\sqrt{1+\frac{1}{(\omega\tau_I)^2}}$$

$$\phi = \tan^{-1}(-\frac{1}{\omega\tau_I}) < 0$$

$$\log(\frac{AR}{K_c}) = \frac{1}{2}\log(1+\frac{1}{(\omega\tau_I)^2})$$

1- **Low frequency asymptote**

as $\omega \to 0\,,\ \frac{1}{(\omega\tau_I)^2} >> 1$ thenn $\log(\frac{AR}{K_c}) \to -\log(\omega\tau_I)$

Consequently, the LFA is a straight line with slope=-1

$$\phi = \tan^{-1} - \frac{1}{0} = -90^{o}$$

2- **High frequency asymptote**

as $\omega \to \infty\,,\ \frac{1}{(\omega\tau_I)^2} \to 0$ then $\to \log(\frac{AR}{K_c}) \to 0$ i.e $\frac{AR}{K_c} \to 1$

HFA is a horizontal line at value $\frac{AR}{K_c} = 1$

For the ϕ

as $\omega \to 0\,,\ \phi \to -90$

as $\omega \to \omega_c\,,\ \phi \to -45$

as $\omega \to \infty\,,\ \phi \to 0$

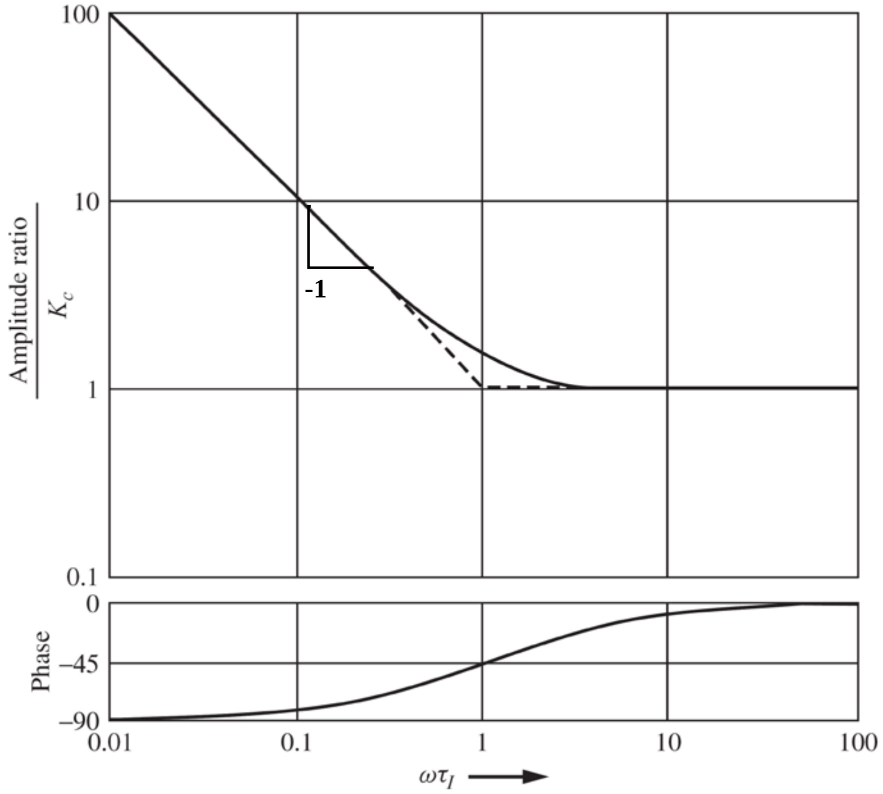


Figure Bode diagram for PI controller.

2-Propertional Derivative controller (PD)

$\mathrm{AR} = \mathrm{K_c}\sqrt{1+\tau_D^2\omega^2}$

$\phi = \tan^{-1}(\omega\tau_D) > 0$

1) **Low frequency asymptote**

as $\omega \to 0$, $\log(\frac{AR}{K_c}) = \frac{1}{2}\log(\tau_D^2\omega^2) = 0 \Rightarrow \frac{AR}{K_c} = 1$ (L.F.A) slope=0

$\phi = \tan^{-1} 0 = 0^o$

2) **High frequency asymptote**

as $\omega \to \infty$, $\log(\frac{AR}{K_c}) = \frac{1}{2}\log(\tau_D^2\omega^2) = \log(\tau_D\omega)$ (H.F.A) slope=+1

$\phi = \tan^{-1} 0 = 90^o$

as $\omega = 0$, $\phi = 0$

as $\omega = \omega_c$, $\phi = +45^o$

as $\omega = \infty$, $\phi \to +90^o$

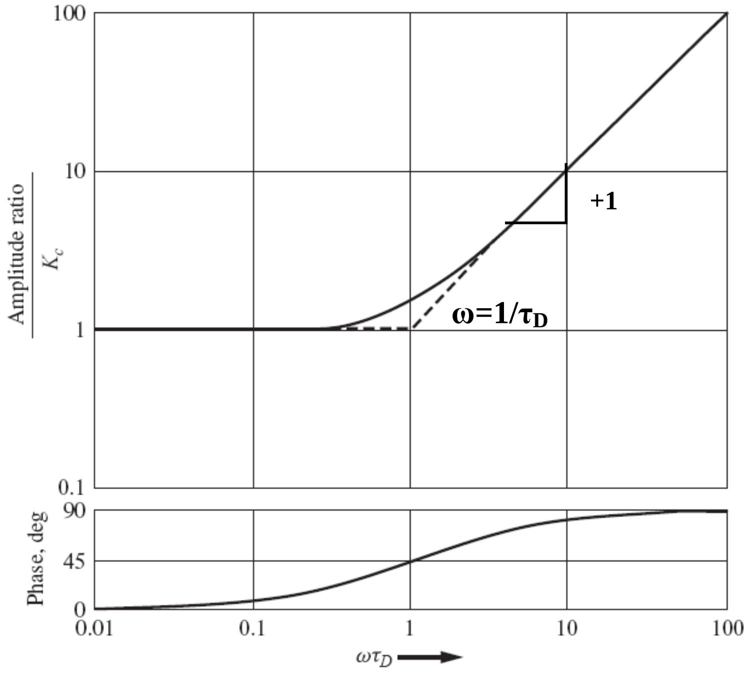


Figure: Bode diagram for PD controller.

3-Propertional Integral Derivative controller (PID)

$$\frac{AR}{Kc}=\sqrt{1+(\tau_D\omega-\frac{1}{\tau_I\omega})^2}$$

$$\phi=\tan^{-1}(\tau_D\omega-\frac{1}{\tau_I\omega})$$

1) as $\omega\to 0$ then $\frac{AR}{Kc}=\sqrt{1+(\frac{1}{\tau_I\omega})^2}$ PI Controller

2) as $\omega\to\infty$ then $\frac{AR}{Kc}=\sqrt{1+(\tau_D\omega)^2}$ PD Controller

3) as $\omega\to\frac{1}{\tau_I}$ then $\frac{AR}{Kc}=\sqrt{1+(\tau_D\omega-1)^2}$

4) as $\omega\to\frac{1}{\tau_D}$ then $\frac{AR}{Kc}=\sqrt{1+(1-\frac{1}{\tau_I\omega})^2}$

Frequency Response of non-interacting capactine in sereies

$$G(s)=G_1(s)\times G_2(s)\times G_3(s)\times ....\times G_n(s)$$

$$AR_t=AR_1\times AR_2\times AR_3\times ....\times AR_n$$

$$\log AR_t=\log AR_1+\log AR_2+\log AR_3+............+\log AR_n$$

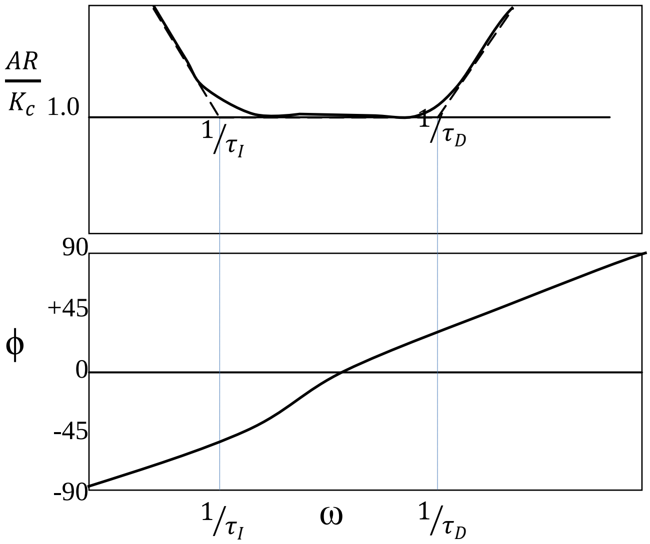
$$\phi_t=\phi_1+\phi_2+\phi_3+............+\phi_n$$

Example: **Bode Digram of PID Controller**

$$G_1(s) = 10(1 + \frac{1}{10s} + 5s)$$

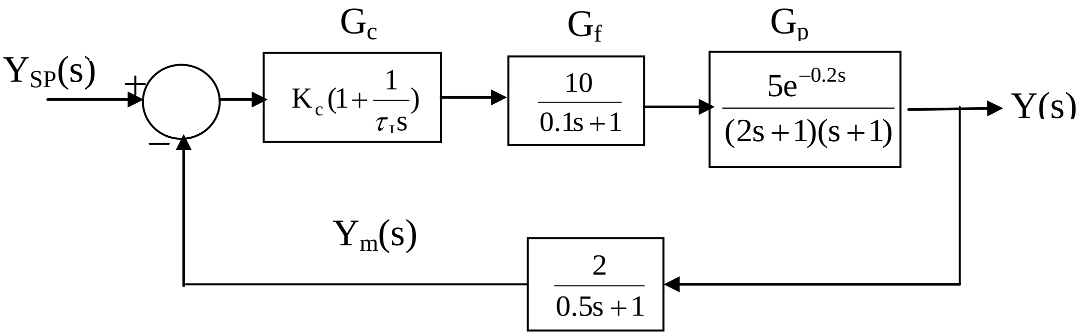
$$\omega_{c1}(s) = \frac{1}{10} = 0.1 \qquad \text{signal}(-1)$$

$$\omega_{c1}(s) = \frac{1}{5} = 0.2 \qquad \text{signal}(+1)$$



Example:

Bode plots of open loop system



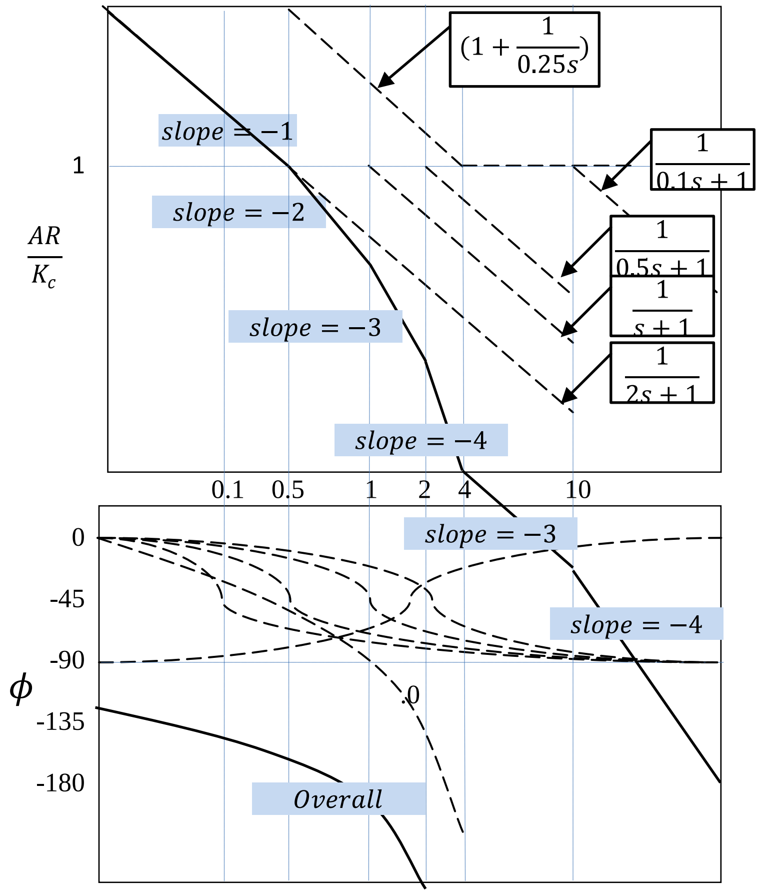
The Open loop T.F. of the feedback control

$$G_{OL} = G_c\, G_f\, G_p\, G_m$$

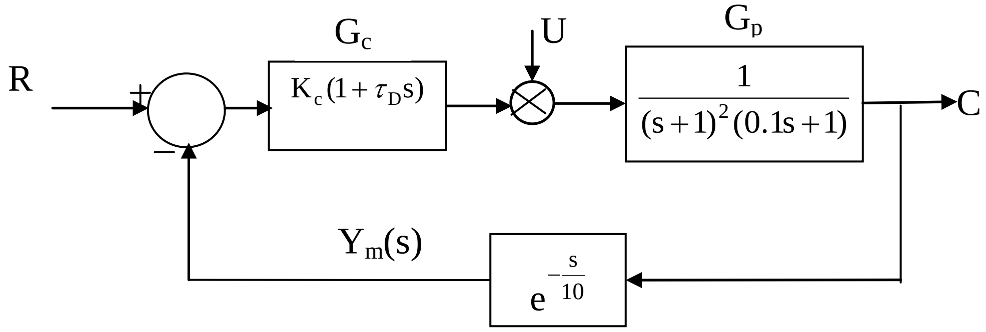
$$G_{OL} = 100K_c(1 + \frac{1}{\tau_I s})\frac{1}{0.1s+1}\cdot\frac{1}{(2s+1)(s+1)}\cdot\frac{1}{(0.5s+1)}e^{-0.2s}$$

With K_c=4 and τ_I=0.25

$\therefore$K=400



Example: Plot the B.D. for the open loop T.F. for the fig. below



For Kc=10 and τ_D=0.5 the overall transfer function is

$$G_{OL}(s)=\frac{10(0.5s+1)e^{-\frac{s}{10}}}{(s+1)^2(0.1s+1)}$$

Overall Bode diagram

$G_1(s)=\frac{1}{1s+1}\rightarrow\omega_{c1}=\frac{1}{1}=1$ **stright line slope=-1**

$G_2(s)=\frac{1}{1s+1}\rightarrow\omega_{c2}=\frac{1}{1}=1$ **stright line slope=-1**

$G_3(s)=0.5s+1\rightarrow\omega_{c3}=\frac{1}{0.5}=2$ **stright line slope=+1**

$G_4(s)=\frac{1}{0.1s+1}\rightarrow\omega_{c4}=\frac{1}{0.1}=10$ **stright line slope=-1**

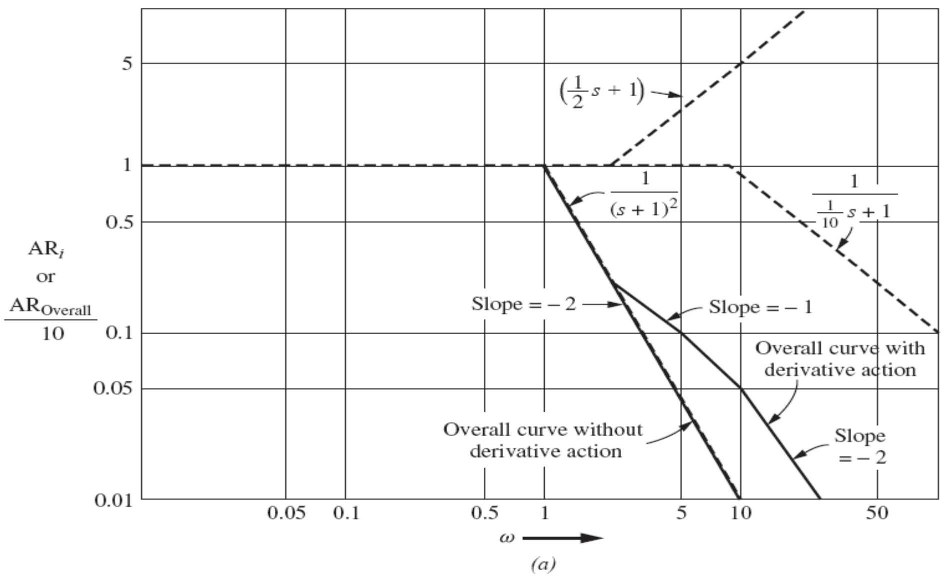
$G_5(s)=e^{-\frac{s}{10}}$ **stright line slope=0**

Amplitude Ratio Curve Prediction

ω	SL1	SL2	SLC	SL3	sLd	SLTotal
0-1	0	0	0	0	0	0
1-2	-1	-1	0	0	0	-2
2-10	-1	-1	+1	0	0	-1
10-	-1	-1	+1	-1	0	-2

Phase Lag Curve Prediction

ω	ϕ L1	ϕL2	ϕ C	ϕ L3	ϕ d	SLTotal
0	0	0	0	0	-	-
1	-45	-45	0	-	-	-
2	-	-	45	-	-	-
10	-	-	-	-45	-	-
∞	-90	-90	90	-90	-∞	-∞



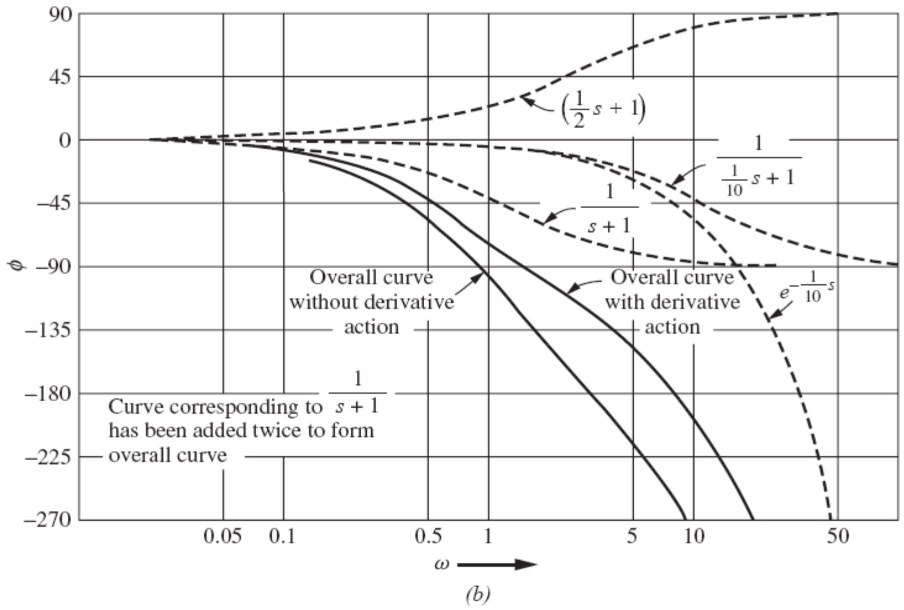
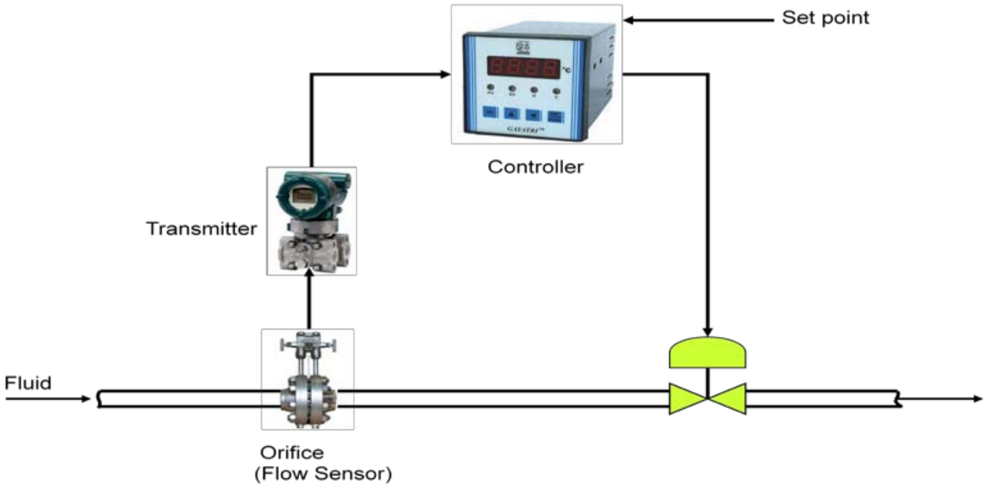


Figure: Block diagram for: (*a*) Amplitude ratio; (*b*) phase angle.



Instrumentation

The example level-control problem had three critical pieces of instrumentation: a sensor (measurement device), actuator (manipulated input device), and controller. The sensor measured the tank level, the actuator changed the flow rate, and the controller determined how much to vary the actuator, based on the sensor signal. Each device in a control loop must supply or receive a signal from another device.

Sensors (Sensing Element)

A device, usually electronic, which detects a variable quantity and measures and converts the measurement into a signal to be recorded elsewhere. A sensor is a device that measures a physical quantity and converts it into a signal which can be read by an observer or by an instrument.

There are many common sensors used for chemical processes. These include temperature, level, pressure, flow, composition, and pH.

For example, a mercury thermometer converts the measured temperature into expansion and contraction of a liquid which can be read on a calibrated glass tube. A thermocouple converts temperature to an output voltage which can be read by a voltmeter.

Control of unit operations

1) **Level Control**

- A level control is needed whenever there is a V/L or L/L interface
- Many smaller vessels are sized based on level control response time

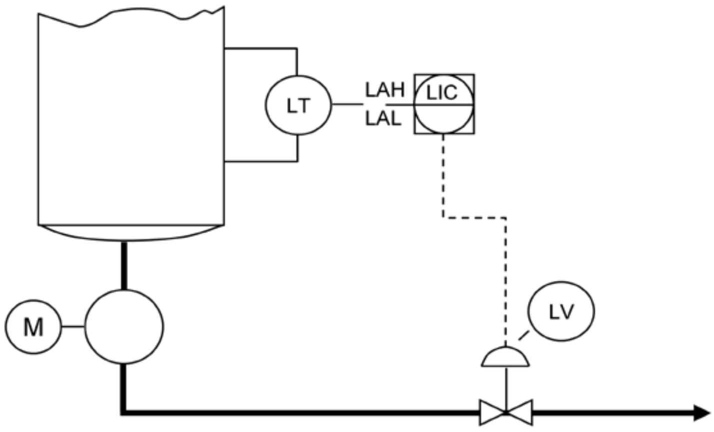


Figure 14.1 Liquid level control system

Example: A boiler drum with a conventional feedback control system is shown in Fig. 14.2. The level of the boiling liquid is measured and used to adjust the feed water flow rate.

This control system tends to be quite sensitive to rapid changes in the disturbance variable, steam flow rate, as a result of the small liquid capacity of the boiler drum.

Rapid disturbance changes can occur as a result of steam demands made by downstream processing units.

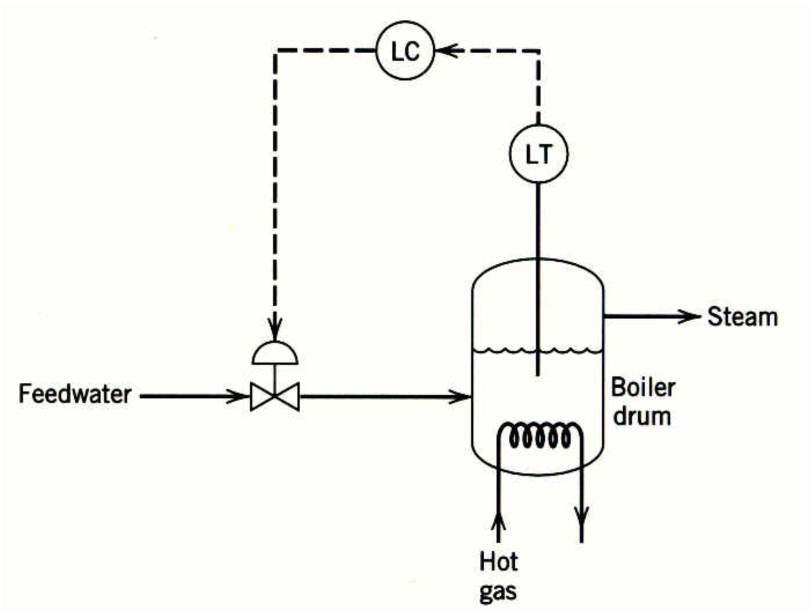


Figure 14.2 The feedback control of the liquid level in a boiler drum.

The feedforward control scheme in Fig. 14.3 can provide better control of the liquid level. Here the steam flow rate is measured, and the feedforward controller adjusts the feedwater flow rate.

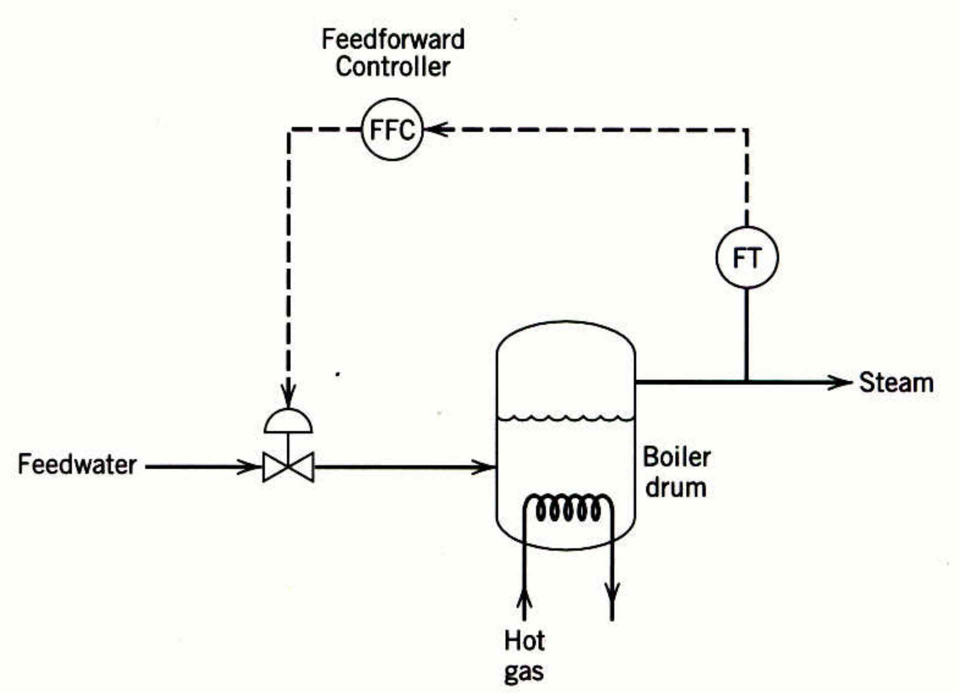


Figure 14.3 The feedforward control of the liquid level in a boiler drum.

2) Pressure Control

- Pressure control is usually by venting a gas or vapor.
- In hydrocarbon processes, off-gas is often vented to fuel.
- In other processes, nitrogen may be brought in to maintain pressure and vented via scrubbers.
- Most common arrangement is direct venting.
- Several vessels that are connected together may have a single pressure controller.

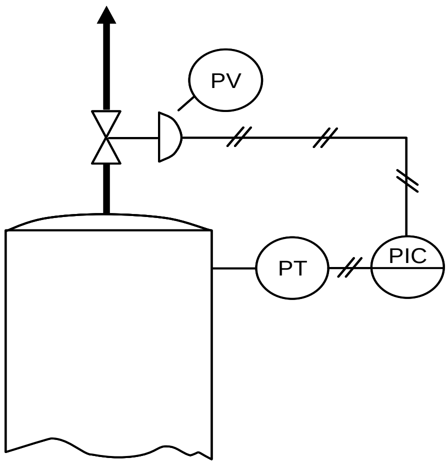


Figure 14.4 Pressure control system

3) Flow Control

- Most common arrangement is a control valve downstream of a pump or compressor.

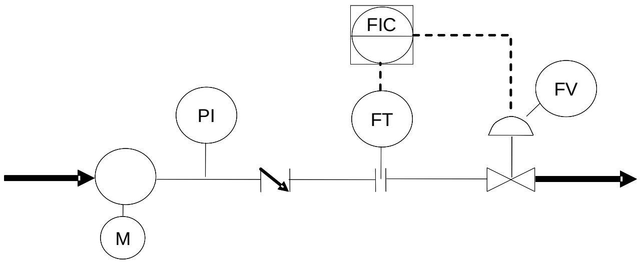


Figure 14.5 Flowrate control system

Example:Vaporizer Flow Control

- Vaporizer flow control needs to prevent liquid accumulation.
- Hence use level controller to actuate heat input to the vaporizer and maintain a constant inventory.
- Control of liquid flow in is easier than control of vapor flow out.

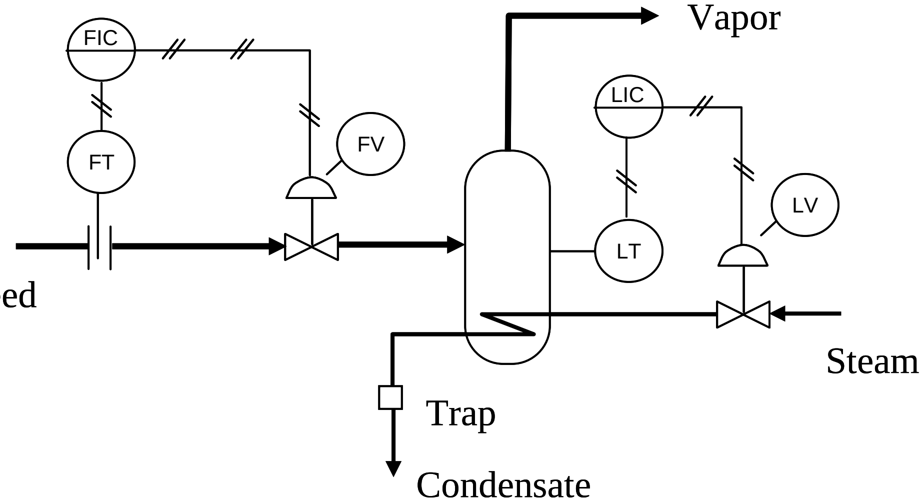


Figure 14.6 Vaporizer control system

4) Temperature Control: Single Stream

- Heaters and coolers are usually controlled by manipulating the flow rate of the hot or cold utility stream.
- Final control element can be on inlet or outlet of utility side.

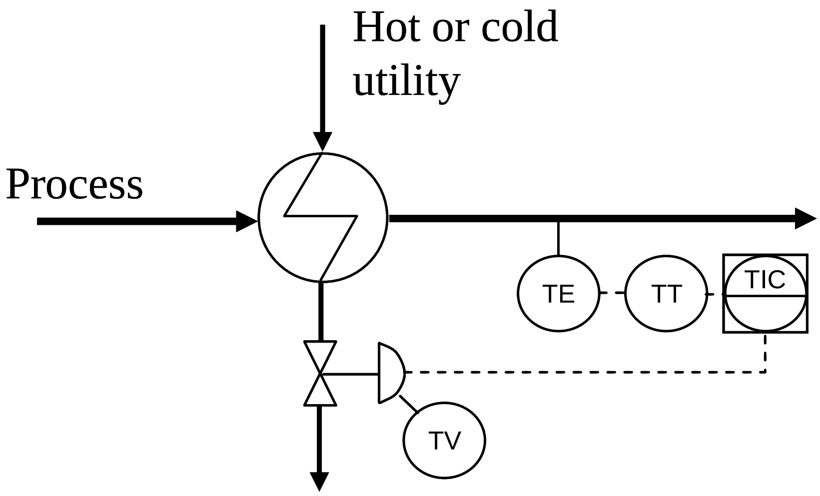


Figure 14.7 Temperature control system

Example: Heat exchangers temperature control

- Temperature control for an heat exchanger is usually by manipulating the flow through a bypass.
- Only one side of an exchanger can be temperature controlled.
- It is also common to see heat exchangers with temperature control on the downstream heater and cooler.

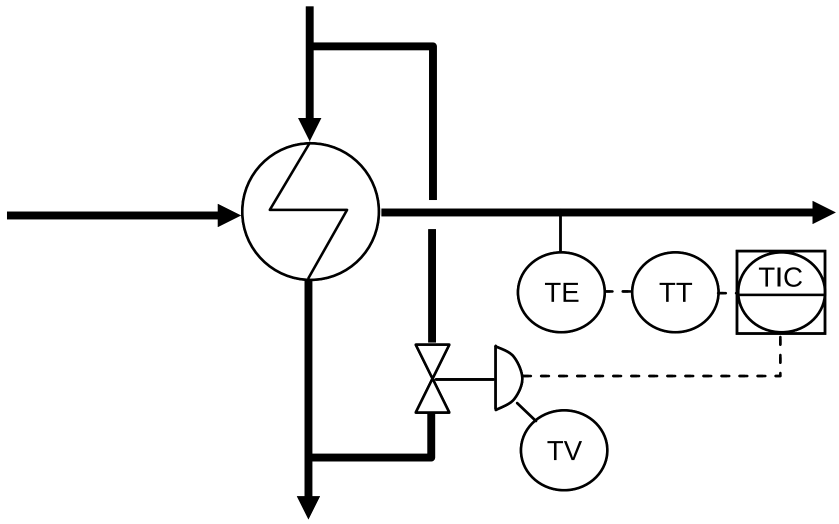


Figure 14.8 Temperature control of heat exchanger

Example: Air coolers temperature control

- Ambient air temperature varies, so air coolers are oversized and controlled by manipulating a bypass.
- Alternatively, air cooler can use a variable speed motor, louvers or variable pitch fans.

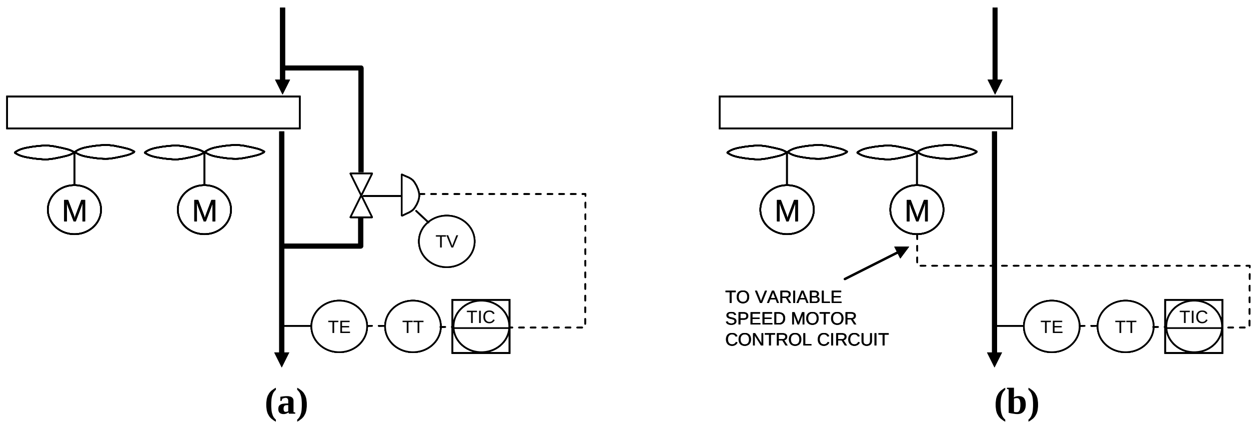


Figure 14.9 Temperature control of air coolers

Example: Temperature Control of CSTR

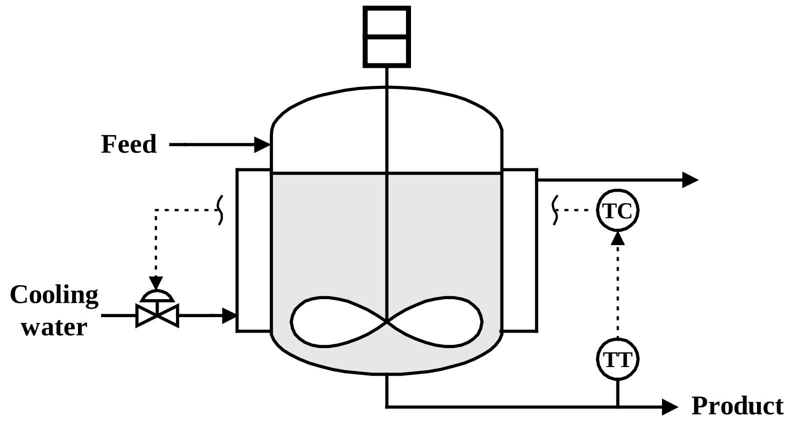


Figure 14.10 Temperature control of CSTR

Distillation Control

- ❖ Distillation control is a specialized subject in its own right.
- ❖ In addition to controlling condenser pressure and level in the sump, a simple distillation column has two degrees of freedom.
  - Material balance (split) and energy balance (heat input or removed).
  - Therefore needs two controllers.
  - Therefore has the possibility that the controllers will interact and “fight” each other.
- ❖ Side streams, intermediate condensers & reboilers, pump-arounds, etc. all add extra complexity and degrees of freedom.

The Energy Balance (LQ) Distillation Column Control Structure

The LQ control structure is the most natural control structure for a simple distillation column. This is because the separation in a distillation column occurs due to successive condensation and vaporization of the counter-current vapour and liquid streams flowing through the column. Adjusting the cold reflux, the source of condensation, and the reboiler duty, the source of vaporization, is then a natural choice for regulating the separation achieved in the column. The LQ control structure shown in figure (14.11 a) is thus the most commonly applied distillation control structure. It is also sometimes referred to as an energy balance structure as changing L (cold reflux) or Q alters the energy balance across the column to affect the distillate to bottoms product split.

Material Balance Distillation Column Control Structures

The other control structures are referred to as material balance structures as the product split is directly adjusted by changing the distillate or bottoms stream flow rate. The material balance structures are applied when a level loop for the LQ structure would be ineffective due to a very small product stream (D or B) flow rate. Figure 14.11 b, c and D show Schematics of DQ, LB and DB distillation column control structures. The DQ structure is thus appropriate for columns with very large reflux ratio ($L/D > 4$). The distillate stream flow is then a fraction of the reflux stream so that the reflux drum level cannot be maintained using the distillate. The level must then be controlled using the reflux. The LB structure is appropriate for columns with a small bottoms flow rate compared to the boil-up. The bottoms stream is then not appropriate for level control and the reboiler duty must be used instead. The DB control structure is used very rarely as both D and B cannot be set independently due to the steady state overall material balance constraint. In dynamics however, the control structure may be used when the reflux and reboil are much larger than the distillate and bottoms respectively.

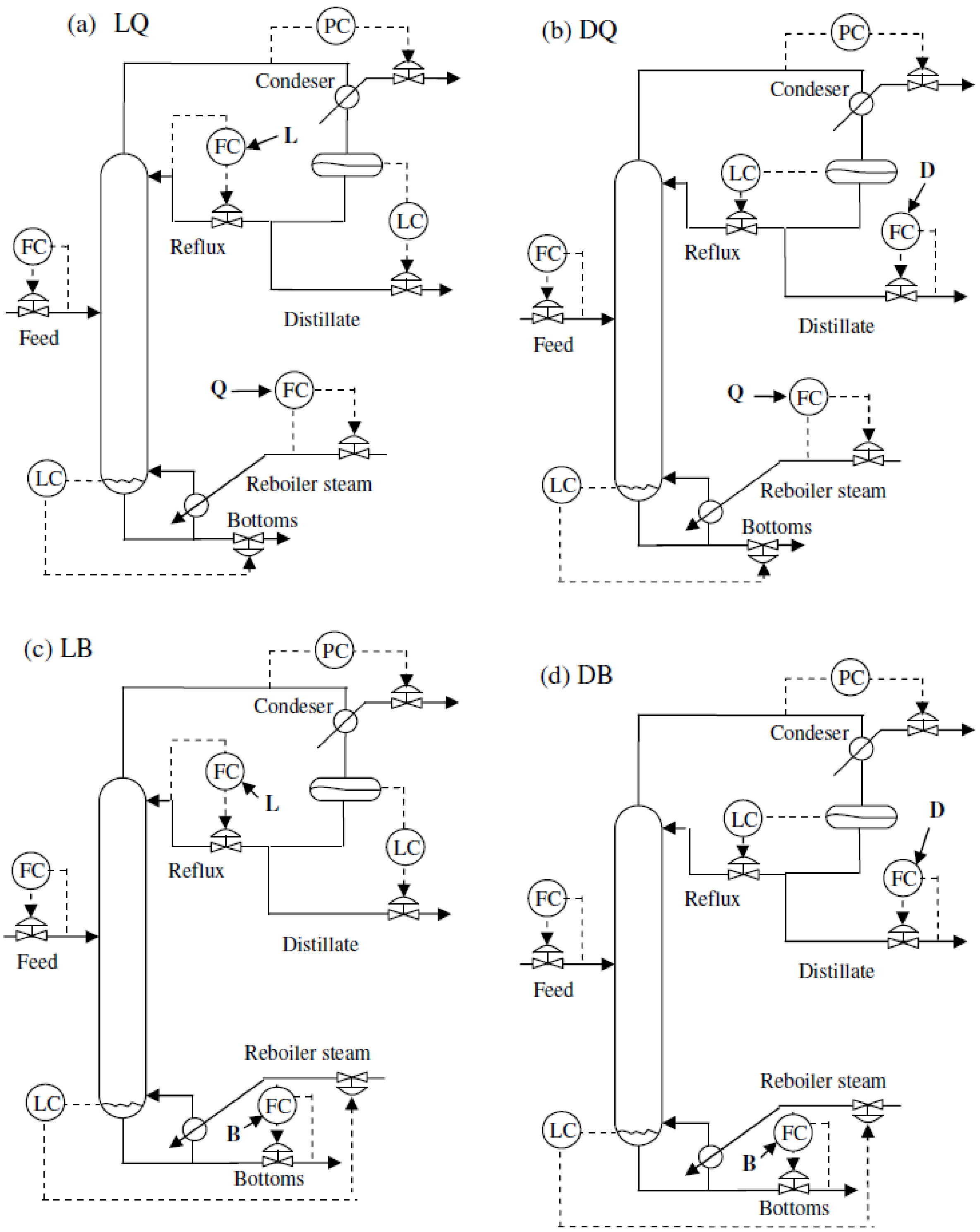


Figure 14.11 Schematics of LQ, DQ, LB and DB distillation column control structures

Other Distillation Column Control Structure

Other variants of the basic control structure types include the L/D-Q, L/D-B and DQ/B. In the first two structures the reflux ratio is adjusted for regulating the separation. In the last structure the reboil ratio is adjusted. These control structures are illustrated in Figure 14.12.

Note that when the reflux is adjusted in ratio with the distillate, the distillate stream can be used to control the reflux drum level even as it may be a trickle compared to the reflux rate.

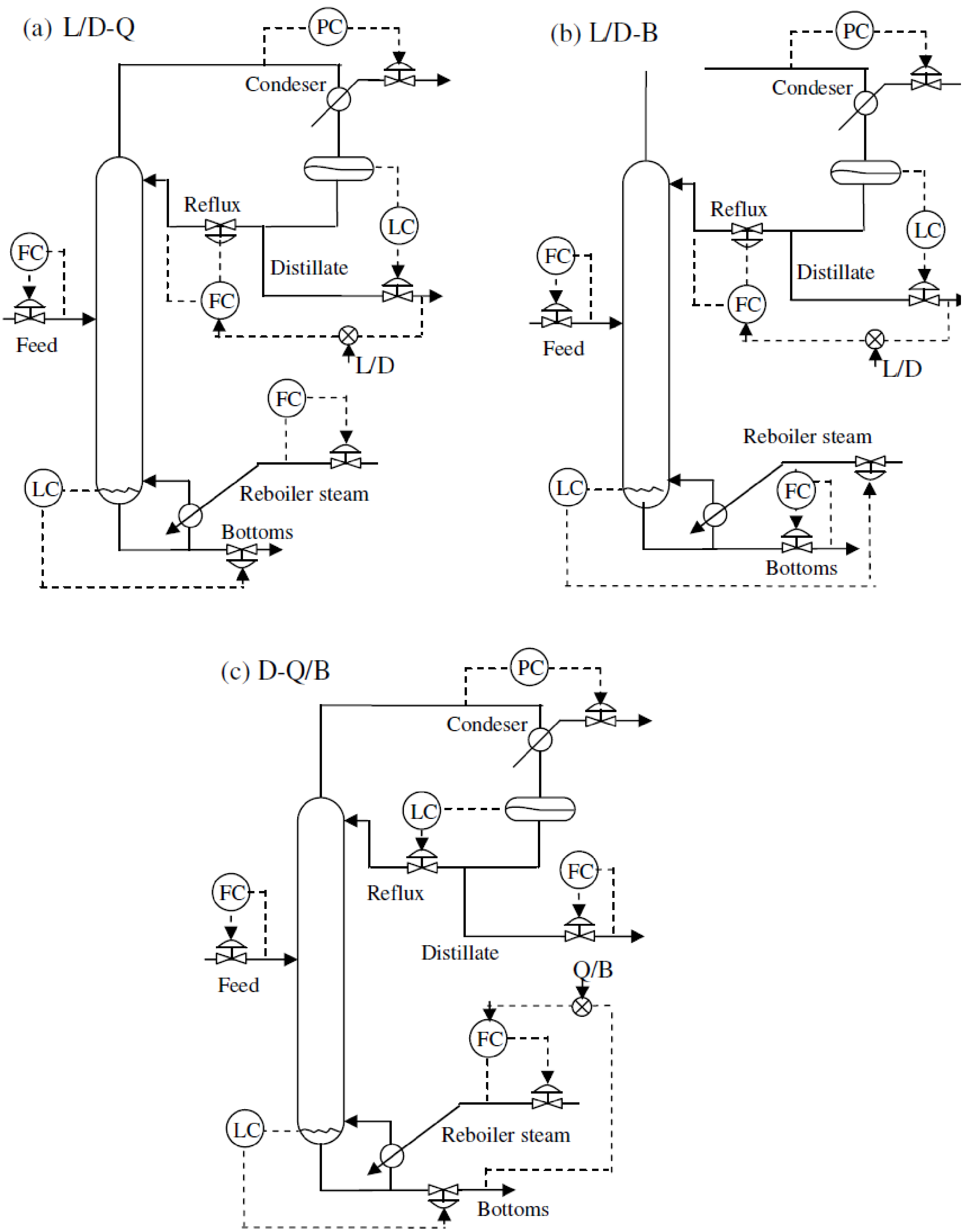


Figure 14.12 Schematics of L/D-Q, L/D-B, and D-Q/B distillation column control structures.

Batch Distillation

- Reflux flow control adjusted based on temperature (used to infer composition)

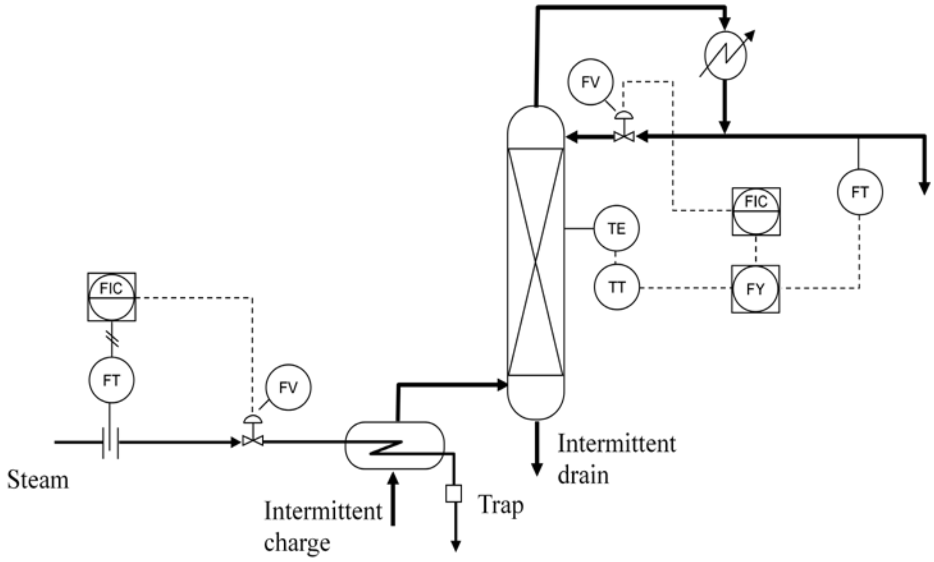
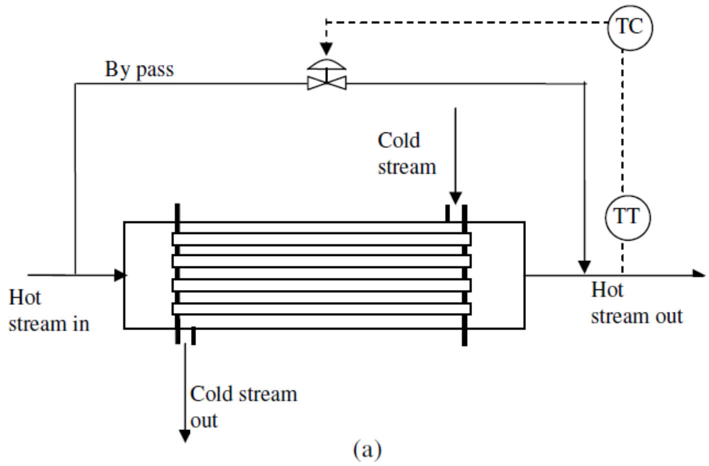


Figure 14.13 Batch distillation column control system

Heat Exchangers

Heat exchangers process used to transfer heat between two process streams. The flow of these process streams is usually set elsewhere in the plant so that adjusting the flowrate of one of the process streams to regulate the amount of heat transferred is not possible.

To provide a control degree-of-freedom for regulating the heat transferred, a small by-pass (~5-10%) of one of the process streams around the heat exchanger is provided. The outlet temperature of this process stream or the other process stream can be controlled by manipulating the by-pass rate. These two schemes are illustrated in Figure 14.14. In the former, tight temperature control is possible as the amount of heat transferred is governed by the bypass. In the latter, a thermal lag of the order of 0.5 to 2 minutes exists between the manipulated and controlled variable.



(a)

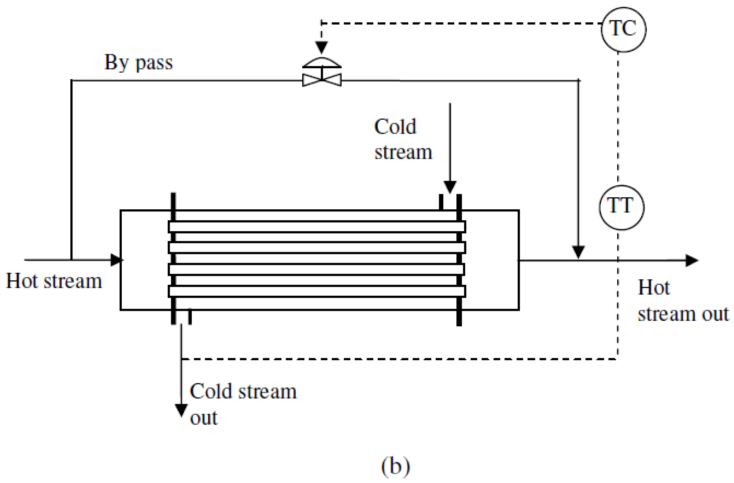


Figure 14.14 By-pass control of process to process heat exchangers
(a) Controlling and bypassing hot stream (b) Controlling cold stream and bypassing hot stream

Control of Miscellaneous Systems

Vapor Absorption Cycle

In addition to compression systems, refrigerant absorption systems are also applied industrially. The absorption based refrigeration cycle and its control scheme is shown in Figure 14.15. Ammonia (refrigerant) rich strong liquor is distilled at high pressure to recover liquid ammonia as the distillate and ammonia lean weak liquor as the bottoms. The liquid ammonia is fed to the evaporator where it absorbs heat from the process stream to be chilled and evaporates. Vapor ammonia is absorbed by the 'weak liquor' water stream. The 'strong liquor' so formed is fed to the distillation column to completed the closed circuit refrigerant loop. The temperature of the chilled process stream is controlled by adjusting the level setpoint of the evaporator. The heat transfer rate is thus varied by changing the area across which heat transfer occurs. The evaporator level controller adjusts the distillate liquid ammonia flow. An increase in the level of the evaporator implies an increase in the ammonia evaporation rate so that the weak liquor rate is increased in ratio to absorb the ammonia vapours. The strong liquor is cooled and collected in a surge drum. The level of the surge drum is not controlled. Liquid from the surge drum is pumped back to the distillation column through a process-to-process heater that recovers heat from the hot 'weak liquor' bottoms from the distillation column. The flow rate of the strong liquor to the column is adjusted to maintain the column bottoms level. Also, the steam to the reboiler is manipulated to maintain a tray temperature.

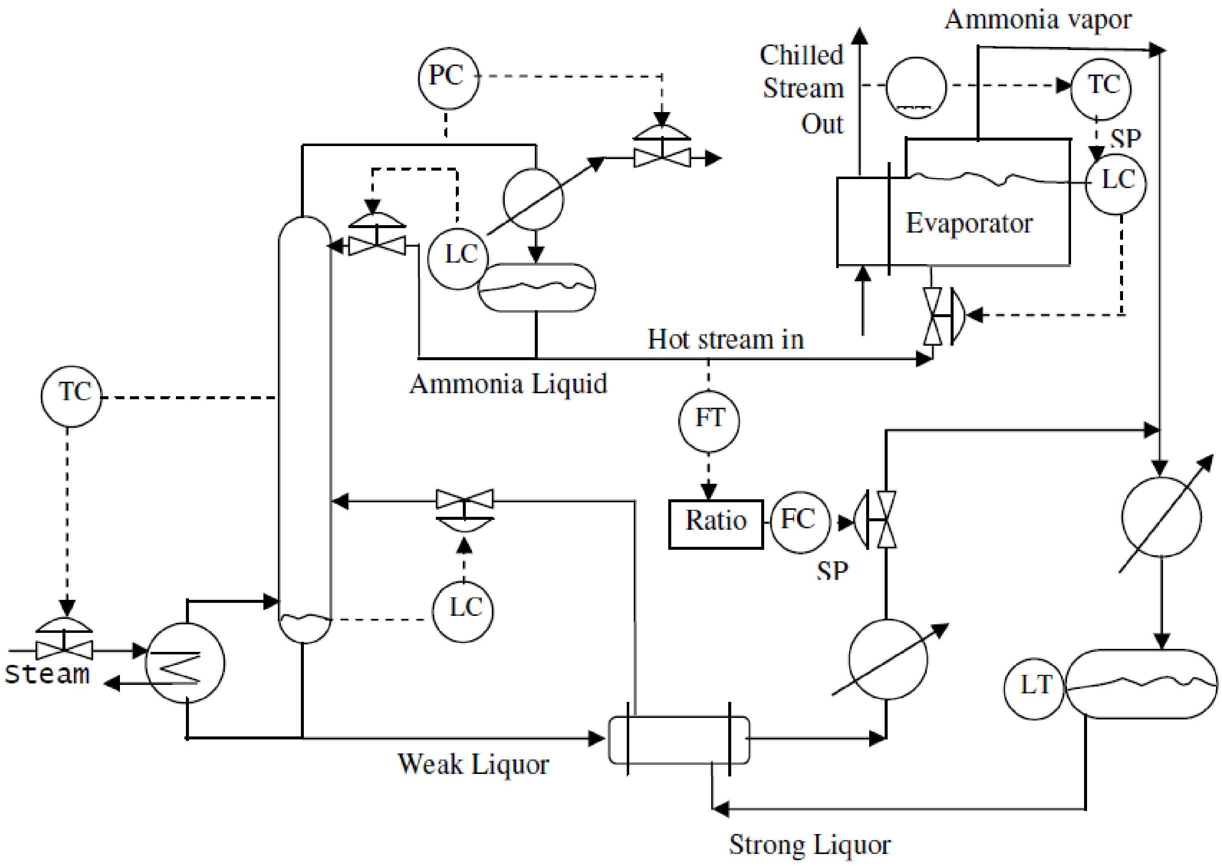


Figure 14.15 Absorption refrigeration control system