Multiple Integrals

Integration of functions of one variable

We start by recalling the basics of integration with respect to a single variable. We have results such as:

$$\int_a^b f(x)\,\mathrm{d}x \;=\; \big[F(x)\big]_a^b \;:=\; F(b)-F(a)$$

where F is an "antiderivative" or indefinite integral of f. ($f = \mathrm{d}F/\mathrm{d}x$.)

$$\int \lambda f(x)\,\mathrm{d}x = \lambda \int f(x)\,\mathrm{d}x,$$

$$\int (f(x)+g(x))\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x\,.$$

Useful techniques include:

- Integration by parts.
- Integration by substitution.
- Use of partial fractions.

Some specific integrals can be found on the formula sheet.

Question 1. Calculate $\displaystyle\int_0^1 \left(3x^4 + x^2\mathrm{e}^{x^3}\right)\mathrm{d}x$.

Solution. This integral is designed to use a few different techniques! Firstly we can split the integral as

$$\int_0^1 \left(3x^4 + x^2\mathrm{e}^{x^3}\right)\mathrm{d}x = 3\int_0^1 x^4\,\mathrm{d}x + \int_0^1 x^2\mathrm{e}^{x^3}\,\mathrm{d}x.$$

The first integral on the right-hand side is straightforward:

$$\int_0^1 x^4\,\mathrm{d}x \;=\; \left[\frac{x^5}{5}\right]_0^1 \;=\; \frac{1}{5}.$$

To calculate the second integral on the right hand side we use integration by substitution; taking $u = x^3$ we have $\mathrm{d}u/\mathrm{d}x = 3x^2$ so $x^2\,\mathrm{d}x = \frac{1}{3}\,\mathrm{d}u$. Therefore

$$\int x^2 \mathrm{e}^{x^3}\,\mathrm{d}x \;=\; \int \frac{1}{3}\mathrm{e}^u\,\mathrm{d}u \;=\; \frac{1}{3}\mathrm{e}^u + C \;=\; \frac{1}{3}\mathrm{e}^{x^3} + C.$$

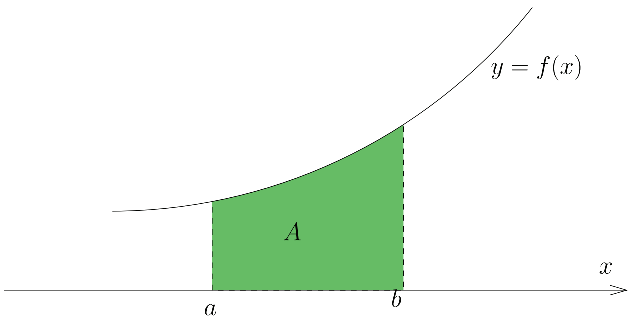
The definite integral can then be computed as

$$\int_0^1 x^2 \mathrm{e}^{x^3}\,\mathrm{d}x \;=\; \left[\frac{1}{3}\mathrm{e}^{x^3}\right]_0^1 \;=\; \frac{1}{3}\mathrm{e}^1 - \frac{1}{3}\mathrm{e}^0 \;=\; \frac{\mathrm{e}-1}{3}.$$

Combining the above calculations we finally get

$$\int_0^1 \left(3x^4 + x^2\mathrm{e}^{x^3}\right)\mathrm{d}x \;=\; 3 \times \frac{1}{5} + \frac{\mathrm{e}-1}{3} \;=\; \frac{4+5\mathrm{e}}{15}.$$

One interpretation of an integral like this is the area under a graph: assuming that f is positive, the area of a strip of width $\mathrm{d}x$ and length f is $f\ \mathrm{d}x$ and then $A = \int_a^b f(x)\,\mathrm{d}x$ is the total area between the x axis and the curve $y = f(x)$ for $a < x < b$.

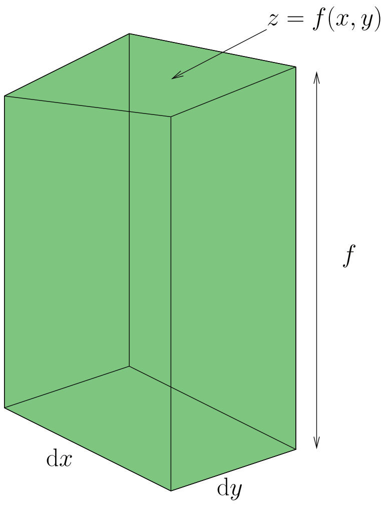


More generally, if f represents the density of some quantity, *i.e.* it is the amount per unit length, $f\ \mathrm{d}x$ is the amount in a short length $\mathrm{d}x$ and $A = \int_a^b f(x)\,\mathrm{d}x$ is the total amount between $x = a$ and $x = b$.

2 Integration of functions of two variables

Thinking of a single integral as giving an area, we might ask: what is the volume under a surface $z = f(x, y)$ lying above a rectangle in the x - y plane, $a < x < b$, $c < y < d$? (We again take $f > 0$ for simplicity.)

First look at a small rectangle of length $\mathrm{d}x$ and width $\mathrm{d}y$. Its area is $\mathrm{d}A = \mathrm{d}x\ \mathrm{d}y$. Then the volume between that small rectangle and the surface is the height of the enclosed (aproximately) cuboidal region times the area: $f\ \mathrm{d}A = f\ \mathrm{d}x\ \mathrm{d}y$.



The volume between the x - y plane and the surface in the slab of width dx between $y = c$ and $y = d$ is then given by integrating with respect to y: $(\int_c^d f(x, y)\,\mathrm{d}y)\,\mathrm{d}x$. (In doing this integral, x is held fixed.) Finally, to get the total volume, we must integrate with respect to x from a to b:

$$\text{volume } = V = \int_a^b \left(\int_c^d f(x, y)\,\mathrm{d}y \right) \mathrm{d}x\,. \tag{7.1}$$

We usually abbreviate this double integral as

$$V = \int_a^b \int_c^d f(x, y)\,\mathrm{d}y\,\mathrm{d}x\,. \tag{7.2}$$

Noting that, for cases we'll be considering at least, we could have integrated with respect to x first and then y, we also have

$$V = \int_c^d \int_a^b f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int\int_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y\,, \tag{7.3}$$

where D is our region of integration, here D is the rectangle $a < x < b$, $c < y < d$. More generally if f once again represents some sort of density, *e.g.* mass per unit area in a sheet of metal, the double integral $\int\int_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y$ is the total amount in the region D.

Question 2. Calculate $\displaystyle\int_2^4 \int_1^3 (xy^2 + y)\,\mathrm{d}y\,\mathrm{d}x$.

Solution. Calculating the inner integral first gives

$$\int_1^3 (xy^2 + y)\,\mathrm{d}y = \left[\frac{xy^3}{3} + \frac{y^2}{2} \right]_1^3 = \left(9x + \frac{9}{2} \right) - \left(\frac{x}{3} + \frac{1}{2} \right) = \frac{26x}{3} + 4\,.$$

Therefore

$$\int_2^4\int_1^3 (xy^2+y)\,\mathrm{d}y\,\mathrm{d}x = \int_2^4 \Big(\frac{26x}{3}+4\Big)\,\mathrm{d}x$$

$$= \left[\frac{13x^2}{3}+4x\right]_2^4 = \Big(\frac{13\times 16}{3}+16\Big) - \Big(\frac{13\times 4}{3}+8\Big) = 13\times 4+8 = 60\,.$$

Question 3. Calculate $\int_0^1\int_2^3 2xy\,\mathrm{d}x\,\mathrm{d}y$.

Solution. Calculating the inner integral first gives

$$\int_2^3 2xy\,\mathrm{d}x \;=\; \big[x^2y\big]_2^3 \;=\; 9y-4y \;=\; 5y.$$

Therefore

$$\int_0^1\int_2^3 2xy\,\mathrm{d}x\,\mathrm{d}y \;=\; \int_0^1 5y\,\mathrm{d}y \;=\; \left[\frac{5y^2}{2}\right]_0^1 \;=\; \frac{5}{2}.$$

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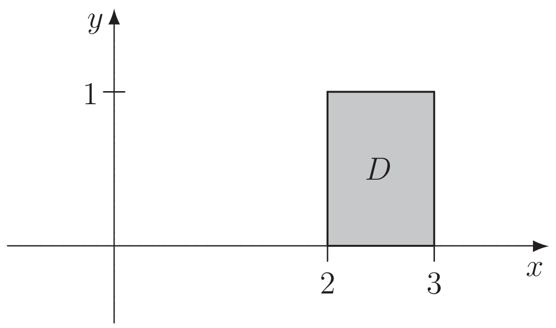
question 4 . For the double integral $\int_2^3\int_0^1 2xy\mathrm{e}^{xy^2}\,\mathrm{d}y\,\mathrm{d}x$ the region of integration D is the rectangle given by

$$0\le y\le 1,\quad 2\le x\le 3.$$



Question 5 . Calculate $\int_2^3\int_0^1 2xy\mathrm{e}^{xy^2}\,\mathrm{d}y\,\mathrm{d}x$.

Solution. Firstly consider the inner integral

$$\int_0^1 2xy\mathrm{e}^{xy^2}\,\mathrm{d}y.$$

We will use integration by substitution for this integral; taking $u=xy^2$ we have $\dfrac{\partial u}{\partial y}=2xy$ so $\mathrm{d}u=2xy\,\mathrm{d}y$ and hence

$$\int 2xy\mathrm{e}^{xy^2}\,\mathrm{d}y \;=\; \int \mathrm{e}^u\,\mathrm{d}u \;=\mathrm{e}^u+C \;=\; \mathrm{e}^{xy^2}+C.$$

Thus

$$\int_0^1 2xye^{xy^2}\,dy \;=\; \left[e^{xy^2}\right]_0^1 = e^x - 1.$$

It follows that

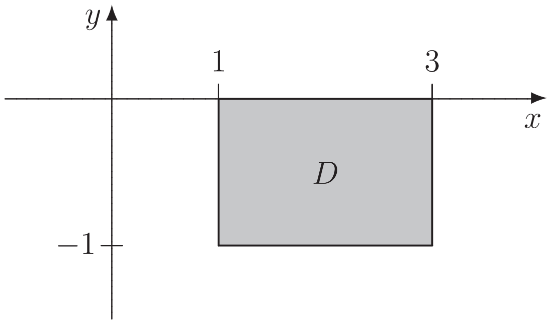
$$\int_2^3\int_0^1 2xye^{xy^2}\,dy\,dx = \int_2^3 (e^x - 1)\,dx = \left[e^x - x\right]_2^3 = (e^3 - 3) - (e^2 - 2) = e^3 - e^2 - 1\,.$$

Question 6 . Let D be the rectangle given by

$$-1 \le y \le 0, \quad 1 \le x \le 3.$$

Calculate $\iint_D (x^2 + y)\,dy\,dx$.

Solution. The region of integration is the following:



Now

$$\iint_D (x^2+y)\,dy\,dx = \int_1^3\int_{-1}^0 (x^2+y)\,dy\,dx = \int_1^3 \left[x^2y + \frac{y^2}{2}\right]_{-1}^0 dx = \int_1^3 \left(x^2 - \frac{1}{2}\right) dx$$

$$= \left[\frac{x^3}{3} - \frac{x}{2}\right]_1^3 = \left(9 - \frac{3}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right) = \frac{23}{3}\,.$$

Of course a region of integration need not be rectangular. If D can be described by $g(x) < y < h(x)$ for $a < x < b$, the (double) integral of $f(x, y)$ over D will be

$$\int\int_D f(x,y)\,dy\,dx = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\,dy\right) dx\,.$$

We can drop the brackets and simply write this as $\int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$.

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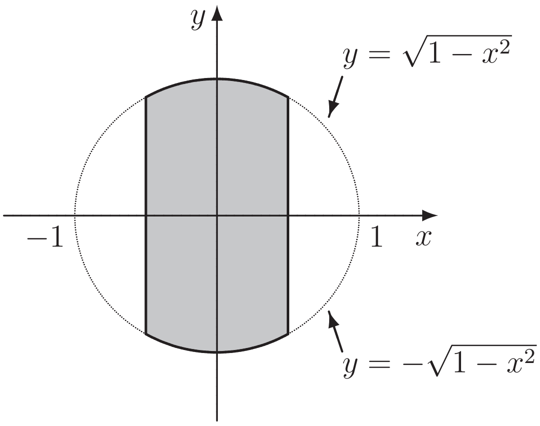
question 7 . Consider the region given by

$$-\frac{1}{2} \le x \le \frac{1}{2}, \quad -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}.$$

This region is part of a disc.



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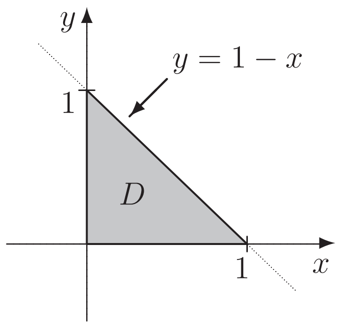
question 8. Consider the region D:



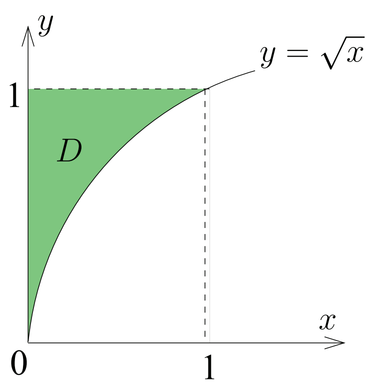
In this region we have $0 \leq x \leq 1$ whilst, for a given x, $0 \leq y \leq 1 - x$. Thus the region D is described by

$$0 \leq x \leq 1, \quad 0 \leq y \leq 1 - x.$$

Question 9 . Calculate $\int\int_D y\,\mathrm{d}A$ over the region D below:

Solution. In this region we have $0 \leq x \leq 1$, whilst, for a given x, we have

$$\sqrt{x} \leq y \leq 1.$$



Therefore

$$\int\int_D y\,dA = \int_0^1 \int_{\sqrt{x}}^1 y\,\mathrm{d}y\,\mathrm{d}x.$$

Calculating the inner integral first gives

$$\int_{\sqrt{x}}^1 y\,\mathrm{d}y \;=\; \left[\frac{y^2}{2}\right]_{\sqrt{x}}^1 \;=\; \frac{1}{2} - \frac{x}{2}.$$

Hence

$$\int_D y\,\mathrm{d}A = \int_0^1 \left(\frac{1}{2} - \frac{x}{2}\right)\mathrm{d}x = \left[\frac{x}{2} - \frac{x^2}{4}\right]_0^1 = 0 - \left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{4}\,.$$

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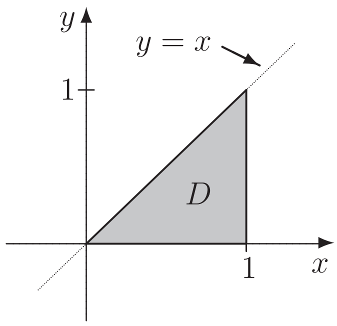
question 10. Consider the region D:



In this region we have $0 \leq y \leq 1$, whilst, for a given y, $y \leq x \leq 1$.

Question 1 1 Calculate $\iint_D (3 - x - y)\,\mathrm{d}A$ where D is the region described in the previous question.

Solution. Recall that D is described by

$$0 \leq y \leq 1, \quad y \leq x \leq 1.$$

Therefore

$$\iint (3 - x - y)\,\mathrm{d}A = \int_0^1 \int_y^1 (3 - x - y)\,\mathrm{d}x\,\mathrm{d}y.$$

Now

$$\int_y^1 (3 - x - y)\,\mathrm{d}x = \left[3x - \frac{x^2}{2} - yx\right]_y^1 = \left(3 - \frac{1}{2} - y\right) - \left(3y - \frac{y^2}{2} - y^2\right) = \frac{5}{2} - 4y + \frac{3y^2}{2}\,.$$

It follows that

$$\iint (3 - x - y)\,\mathrm{d}A = \int_0^1 \left(\frac{5}{2} - 4y + \frac{3y^2}{2}\right)\mathrm{d}y = \left[\frac{5y}{2} - 2y^2 + \frac{y^3}{2}\right]_0^1 = \left(\frac{5}{2} - 2 + \frac{1}{2}\right) - 0 = 1\,.$$

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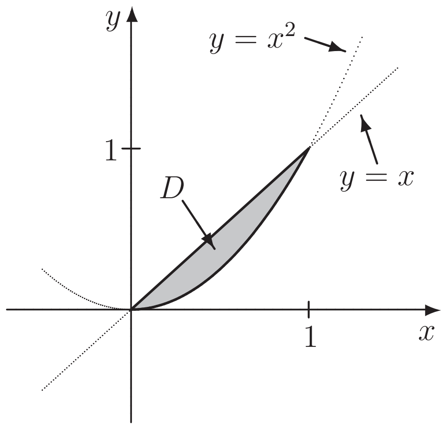
3 Interchanging the order of integration

As noted before, we can swap the order in which the order in which the integrals are carried out : $\int\int_D f\,\mathrm{d}y\,\mathrm{d}x = \int\int_D f\,\mathrm{d}A = \int\int_D f\,\mathrm{d}x\,\mathrm{d}y$. It is sometimes easier to calculate the value of a double integral doing the integrations in one order than the other.

Example 7.6. Consider the region D which lies between the line $y = x$ and the parabola $y = x^2$:



This region can be described by

$$0 \le x \le 1, \quad x^2 \le y \le x.$$

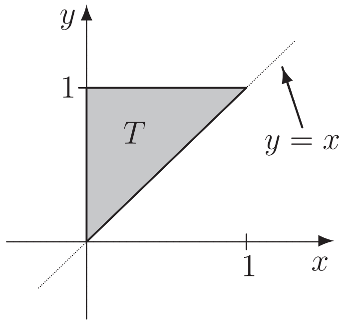
On the other hand, the parabola is also given by the equation $x = \sqrt{y}$ so the region D can also be described by

$$0 \le y \le 1, \quad y \le x \le \sqrt{y}.$$

It follows that we have

$$\int_0^1 \int_{x^2}^{x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x \;=\; \iint_D f(x,y)\,\mathrm{d}A \;=\; \int_0^1 \int_y^{\sqrt{y}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Question 1 . Calculate $\displaystyle\iint_T \mathrm{e}^{y^2}\,\mathrm{d}A$ where T is the triangular region with vertices $(0,0)$, $(0,1)$ and $(1,1)$:



Solution. As a first attempt we may describe T by

$$0 \le x \le 1, \quad x \le y \le 1.$$

It follows that

$$\iint_T \mathrm{e}^{y^2}\,\mathrm{d}A = \int_0^1 \int_x^1 \mathrm{e}^{y^2}\,\mathrm{d}y\,\mathrm{d}x.$$

The inner integral is then

$$\int_x^1 \mathrm{e}^{y^2}\,\mathrm{d}y$$

which can't be evaluated very easily! We've got stuck!

As a second attempt let us describe T the other way; that is

$$0 \le y \le 1, \quad 0 \le x \le y.$$

Then

$$\iint_T \mathrm{e}^{y^2}\,\mathrm{d}A = \int_0^1 \int_0^y \mathrm{e}^{y^2}\,\mathrm{d}x\,\mathrm{d}y.$$

Evaluating the inner integral first gives

$$\int_0^y \mathrm{e}^{y^2}\,\mathrm{d}x \;=\; \left[x\mathrm{e}^{y^2}\right]_0^y \;=\; y\mathrm{e}^{y^2}.$$

It follows that

$$\iint_T \mathrm{e}^{y^2}\,\mathrm{d}A = \int y\mathrm{e}^{y^2}\,\mathrm{d}y = \left[\frac{\mathrm{e}^{y^2}}{2}\right]_0^1 = \frac{\mathrm{e}}{2} - \frac{1}{2} = \frac{1}{2}(\mathrm{e}-1)\,,$$

where we used the substitution $u = y^2$ to evaluate the integral.

Note. On evaluating a double integral $I = \int\int_D f(x,y)\,\mathrm{d}A$ by doing the y integral first, *i.e.* taking $I = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$, we integrate along vertical strips between the lower boundary, say $y = g_1(x)$, and the upper boundary, say $y = g_2(x)$ (this gives a result which generally depends upon x, but definitely does not depend on y): $I_y(x) = \int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y$. We then total up the contributions of strips by integrating I_y from the lowest value of x taken in D, say a, to the largest, say b (see Fig. 7.1(i)): $I = \int_a^b f(x,y)\,\mathrm{d}x$. On the other hand, doing the y integral first, we take $I = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$, we integrate along horizontal strips between the left-hand boundary, say $x = h_1(y)$, and the right-hand boundary, say $x = h_2(y)$ (this gives a result which generally depends upon y, but definitely does not depend on x): $I_x(y) = \int_{h_1(y)}^{h_2(y)} f(x,y)\,\mathrm{d}x$. We then total up the contributions of strips by integrating I_x from the lowest value of y taken in D, say c, to the largest, say d (see Fig. 7.1(ii)). (Either way, the final result does not depend on either x or y!)

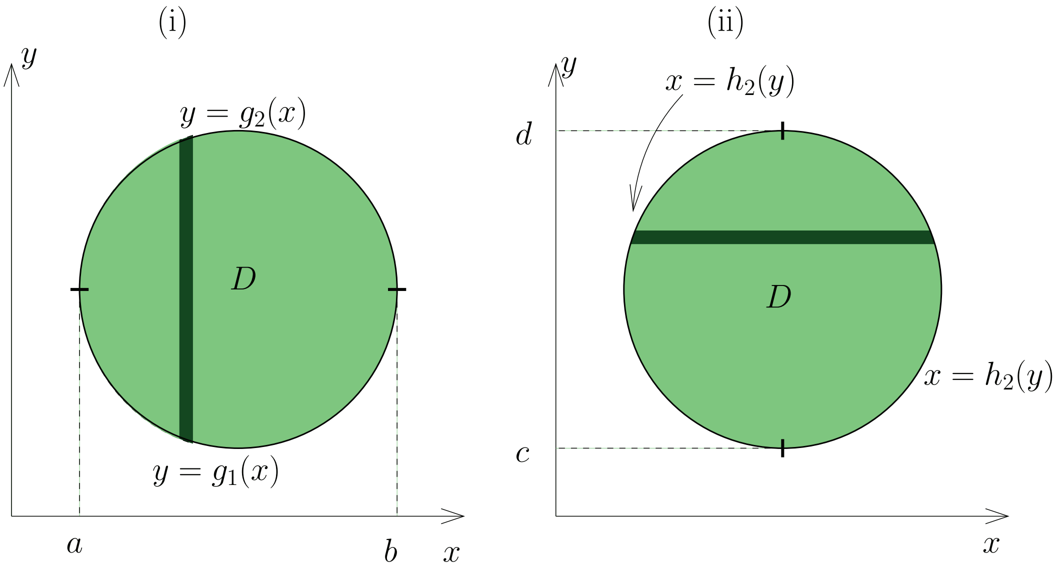


Figure 1: Different orders of integration over D.