Multiple Integrals

① Double integrals over rectangle

if $f(x,y)$ is continous through the rectangular region $R: a \le x \le b$, $c \le y \le d$ then the double integrals over R are defined by

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Example : $\displaystyle\int_0^{3/2}\int^{4-2u} \frac{4-2u}{v^2}\,dv\,du$

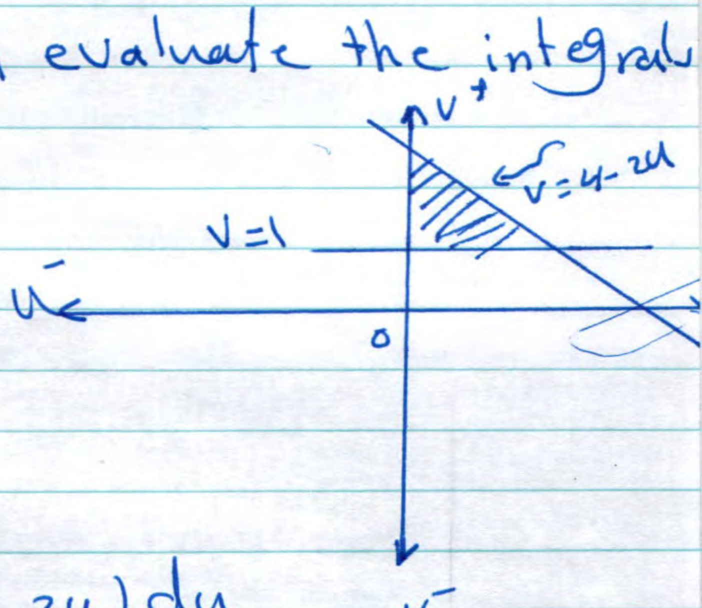
Sketch the region of integration and evaluate the integrals

$$= \int_0^{3/2} (4-2u)\left[-\frac{1}{v}\right]_1^{4-2u} du$$

$$= \int_0^{3/2} (4-2u)\left[-\frac{1}{4-2u}+1\right]$$

$$= \int_0^{3/2} (-1+4-2u)\,du = \int_0^{3/2} (3-2u)\,du$$

$$= \left[3u-u^2\right]_0^{3/2} = \frac{9}{2}-\frac{9}{4} = \frac{9}{4}$$



Example (2)

calculate $\iint_R f(x,y)\,dA$ for $f(x,y) = 1 - 6x^2y$

and $R: 0 \le x \le 2, \; -1 \le y \le 1$

Solution :-

$$\iint_R f(x,y)\,dA = \int_{-1}^{1}\int_{0}^{2}(1-6x^2y)\,dx\,dy$$

$$= \int_{-1}^{1} \left(x - 2x^3y\right)\Big|_0^2 dy$$

$$= \int_{-1}^{1} (2-16y)\,dy = \left[2y - 8y^2\right]_{-1}^{1} = \boxed{4}$$

Reversing the order of integration gives the same answer.

$$\int_0^2\int_{-1}^{1}(1-6x^2y)\,dy\,dx = \int_0^2 \left(y - 3x^2y^2\right)\Big|_{-1}^{1} dx$$

$$= \int_0^2 (1-3x^2) - (-1-3x^2)\,dx$$

$$= \int_0^2 2\,dx = 2x\Big|_0^2 = \boxed{4}$$

Example ③

$$\iint_R x y e^{xy^2} \, dA \quad , R: 0 \leq x \leq 2$$
$$0 \leq y \leq 1$$

Solution : $\int_0^2 \int_0^1 x y e^{xy^2} \, dy \, dx$

$t = y^2$
$dt = 2y \, dy$

$$= \frac{1}{2} \int_0^2 \int_0^1 x e^{xt} \, dt \, dx$$

$$= \frac{1}{2} \int_0^2 x \cdot \frac{e^{xt}}{x} \Big|_0^1 \, dx$$

$$= \frac{1}{2} \int_0^2 (e^x - 1) \, dx$$

$$= \frac{1}{2} \left[e^x - x \right]_0^2$$

$$= \frac{1}{2} \left[e^2 - 2 - 1 + 0 \right] = \frac{1}{2} (e^2 - 3)$$

Example 4 : calculate $\int_2^4\int_1^3 (xy^2+y)\,dy\,dx$

Solution :- $\int_1^3 (xy^2+y)\,dy = \left[\frac{xy^3}{3}+\frac{y^2}{2}\right]_1^3$

$$= \left(9x+\frac{9}{2}\right) - \left(\frac{x}{3}+\frac{1}{2}\right)$$

$$= \frac{26x}{3} + 4$$

Therefore $\int_2^4\int_1^3 (xy^2+y)\,dy\,dx$

$$= \int_2^4 \left(\frac{26x}{3}+4\right) dx$$

$$= \left[\frac{13x^2}{3}+4x\right]_2^4$$

$$= \left[\frac{13\times 16}{3}+4x\right]_2^4$$

$$= \left(\frac{13\times 16}{3}+16\right) - \left(\frac{13\times 4}{3}+8\right)$$

$$= 13\times 4 + 8 = 60$$

Triple integrals in rectangular Coordinates

the volume V of a closed bounded region D in space is $V = \iiint_D dV$

How to evaluate the triple integrals?

(1) integrate firstly with respect to z.
(2) sketch the projection R of the solid on the xy-plane. from this, determine the limits of integration for the double integrals over R.
(3) Integrate the double integrals as in the previous sections $\iiint_D f(x,y,z)\, dV$

$$= \iint_R \left[\int f(x,y,z)\, dz \right] dA$$

(*) Defintion

The average value of a function $f(x,y,z)$ over a region D in space is defined by

$$av(f) = \frac{1}{\text{Volume of } D} \iiint_D f(x,y,z)\, dV$$

Example 1 :- Evaluate $\int_0^1\int_0^{1-x^2}\int_3^{4-x^2-y} x\,dz\,dy\,dx$

$$= \int_0^1\int_0^{1-x^2} x\,z\Big|_3^{4-x^2-y} dy\,dx = \int_0^1\int_0^{1-x^2} x\,(4-x^2-y-3)\,dy\,dx$$

$$= \int_0^1\int_0^{1-x^2} (x-x^3-xy)\,dy\,dx = \int_0^1 \Big[xy - x^3y - x\frac{y^2}{2}\Big]_0^{1-x^2}$$

$$= \int_0^1 x(1-x^2) - x^3(1-x^2) - \frac{1}{2}x(1-x^2)^2\Big]\,dx$$

$$= \int_0^1 \Big[x - \cancel{x^3} - x^3 + x^5 - \frac{1}{2}x + \cancel{x^3} - \frac{1}{2}x^5\Big]\,dx$$

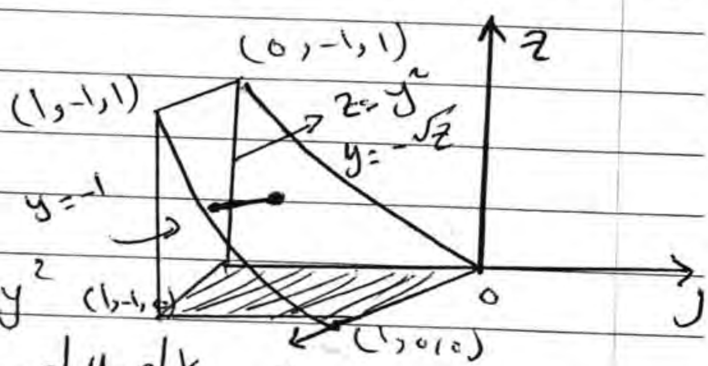
$$= \int_0^1 \Big[\frac{1}{2}x - x^3 + \frac{1}{2}x^5\Big]\,dx$$

$$= \frac{1}{4}x^2 - \frac{1}{4}x^4 + \frac{1}{12}x^6 = \frac{1}{4} - \frac{1}{4} + \frac{1}{12} - 0 = \boxed{\frac{1}{12}}$$

Example 2

: (a) Evaluate $\int_0^1 \int_{-1}^0 \int_0^{y^2} dz\,dy\,dx$

(b) Rewrite the integral as an equivalent integral in the order $dy\,dz\,dx$



(a) Solution $\int_0^1 \int_{-1}^0 z \Big|_0^{y^2} dy\,dx$

$= \int_0^1 \int_{-1}^0 y^2\,dy\,dx = \int_0^1 \frac{y^3}{3} \Big|_{-1}^0 dx = \frac{1}{3} \int_0^1 0-(-1)^3\,dx$

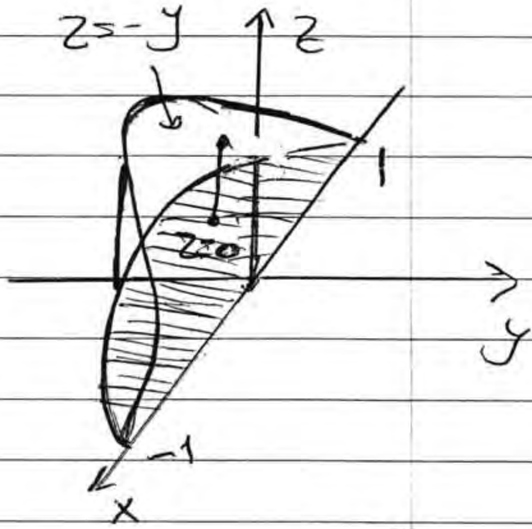
$= \frac{1}{3} \int_0^1 dx = \frac{1}{3} x \Big|_0^1 = \boxed{\frac{1}{3}}$

(b) $\int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy\,dz\,dx = \int_0^1 \int_0^1 y \Big|_{-1}^{-\sqrt{z}} dz\,dx$

$= \int_0^1 \int_0^1 (-\sqrt{z}+1)\,dz\,dx = \int_0^1 \left[-\frac{2}{3} z^{3/2} + z\right]_0^1 dx$

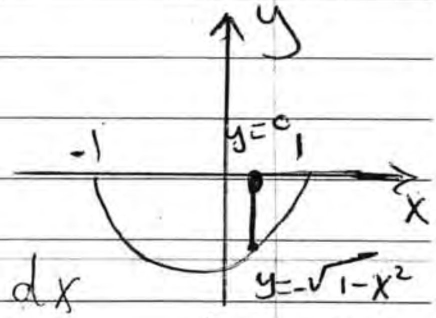
$= \int_0^1 (\frac{-2}{3}+1)\,dx = \int_0^1 \frac{1}{3}\,dx = \frac{1}{3} x \Big|_0^1 = \boxed{\frac{1}{3}}$

(2) Find the volume of the region cut from the cylinder $x^2+y^2=1$ by the planes $z=-y$ and $z=0$



$$V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0}\int_{0}^{-y} dz\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0} -y\,dy\,dx = \int_{-1}^{1} -\frac{y^2}{2}\Big|_{-\sqrt{1-x^2}}^{0} dx$$



$y^2=1-x^2$

$y=-\sqrt{1-x^2}$

$$= -\frac{1}{2}\int\int_{-1}^{1} 0-\left(-\sqrt{1-x^2}\right) dx$$

$$= \frac{1}{2}\int_{-1}^{1}(1-x^2)\,dx = \frac{1}{2} * 2\int_{0}^{1}(1-x^2)\,dx$$

$$= \left[x-\frac{x^3}{3}\right]_0^1 = 1-\frac{1}{3} = \boxed{\frac{2}{3}}$$

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(3) Find the average value of $f(x,y,z) = x+y-z$ over the rectangular solid in the first octant bounded by the coordinate planes and the planes $x=1$, $y=1$, $z=2$

Solution :-

$$V(D) = \int_0^1\int_0^1\int_0^2 dz\,dy\,dx$$

$$= \int_0^1\int_0^1 z\Big|_0^2 dy\,dx$$

$$= \int_0^1\int_0^1 2\,dy\,dx = \int_0^1 2y\Big|_0^1 dx = \int_0^1 2\,dx = 2(x)\Big|_0^1 = 2$$

$$av(f) = \frac{1}{2}\int_0^1\int_0^1\int_0^2 (x+y-z)\,dz\,dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^1 \left(xz+yz-\frac{z^2}{2}\right]_0^2 dy\,dx = \frac{1}{2}\int_0^1\int_0^1 [2x+2y-2]\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 \left[2xy+y^2-2y\right]_0^1 dx = \frac{1}{2}\int_0^1 (2x+1-2)\,dx$$

$$= \frac{1}{2}\int_0^1 (2x-1)\,dx = \frac{1}{2}\left[x^2-x\right]_0^1 = \frac{1}{2}[1-1] = 0$$

(5)

(4) Triple integrals in cylindrical and spherical Coordinates.

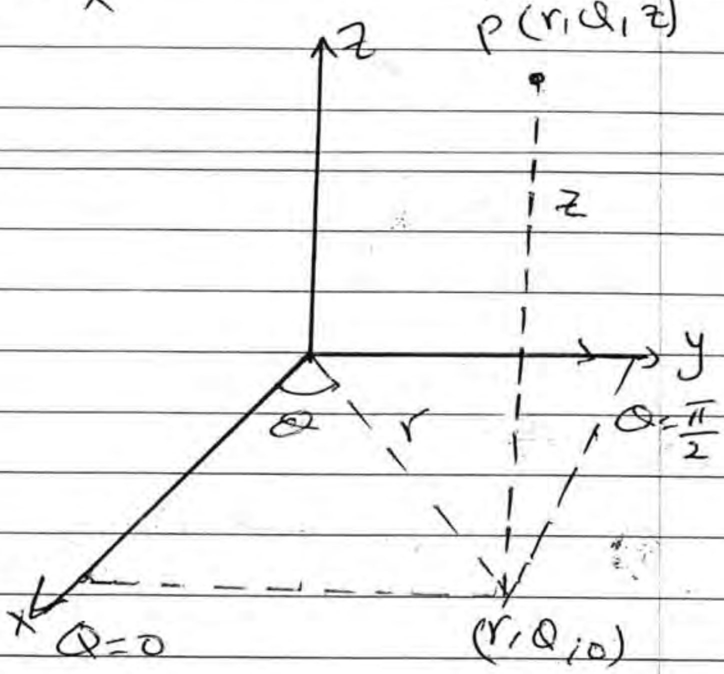
(1) Integration in cylindrical Coordinates

cylindrical Coordinates of a Point P in space is P in space is $P(r, \theta, z)$ in which (i) r and θ are the Polar Coordinates of the vertical Projection of P on the xy_plane.

(ii) z is the rectangular vertical Coordinate, Projection

$x = r\cos\theta$, $y = r\sin\theta$

$x^2 + y^2 = r^2$ and $\tan\theta = \frac{y}{x}$



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