⑥

Example ①: Evaluate: $\int_0^2 \int_{r-2}^{\sqrt{4-r^2}} \int_0^{2\pi} (r\sin\theta+1)\,r\,d\theta\,dz\,dr$

$$= \int_0^2 \int_{r-2}^{\sqrt{4-r^2}} \int_0^{2\pi} (r^2\sin\theta + r)\,d\theta\,dz\,dr$$

$$= \int_0^2 \int_{r-2}^{\sqrt{4-r^2}} \left[-r^2\cos\theta + r\theta\right]_0^{2\pi} dz\,dr$$

$$= \int_0^2 \int_{r-2}^{\sqrt{4-r^2}} \left[-r^2 + 2\pi r + r^2 - 0\right] dz\,dr$$

$$= \int_0^2 \int_{r-2}^{\sqrt{4-r^2}} 2\pi r\,dz\,dr = 2\pi \int_0^2 rz \Big|_{r-2}^{\sqrt{4-r^2}} dr$$

$$= 2\pi \int_0^2 \left[r\sqrt{4-r^2} - r(r-2)\right] dr$$

$$= 2\pi \int_4^0 \left(-\tfrac{1}{2}\right)\sqrt{w}\,dw - 2\pi \int_0^2 (r^2 - 2r)\,dr \qquad \begin{aligned} w &= 4 - r^2 \\ dw &= -2r\,dr \\ -\tfrac{1}{2}dw &= r\,dr \end{aligned}$$

$$= \pi \int_0^4 w^{1/2}\,dw - 2\pi\left[\frac{r^3}{3} - r^2\right]_0^2$$

$$= \pi \frac{w^{3/2}}{3/2}\Big|_0^4 - 2\pi\left[\frac{8}{3} - 4\right]$$

$$= \frac{2}{3}\pi\left[(4)^{3/2} - 0\right] - 2\pi\left(\frac{-4}{3}\right)$$

$$= \frac{2}{3}\pi(8) + \frac{8\pi}{3} = \frac{16\pi}{3} + \frac{8\pi}{3} = \frac{24\pi}{3} = \boxed{8\pi}$$

Class

(5) Convert the integral $\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{0}^{x}(x^2+y^2)\,dz\,dx\,dy$ to an equivalent integral in cylinder Coordinate and evaluate the resulting integral.

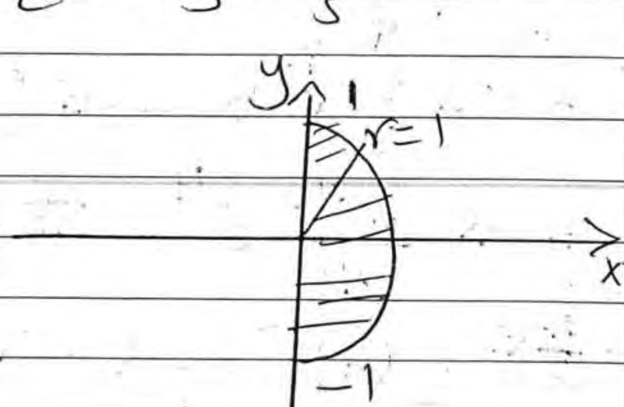
Solution :- $\int_{-\pi/2}^{\pi/2}\int_{0}^{1}\int_{0}^{r\cos\theta} r^2\,dz\,r\,dr\,d\theta$

$$=\int_{-\pi/2}^{\pi/2}\int_{0}^{1} z\Big|_{0}^{r\cos\theta} r^3\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_{0}^{1} r^4\cos\theta\,dr\,d\theta$$

$$=\int_{-\pi/2}^{\pi/2} \frac{r^5}{5}\Big|_{0}^{1}\cos\theta\,d\theta = \frac{1}{5}\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta$$

$$=\frac{2}{5}\int_{0}^{\pi/2}\cos\theta\,d\theta$$

$$=\frac{2}{5}\sin\theta\Big|_{0}^{\pi/2} = \frac{2}{5}[1-0] = \frac{2}{5}$$



(6) Find the volume of the region bounded by the cylinder $x^2+y^2=4$ and the planes $z=0$ and $x+y+z=4$

Solution

$$\int_0^{2\pi}\int_0^{2}\int_0^{4-x-y} dz\, r\, dr\, d\theta$$

$$=\int_0^{2\pi}\int_0^{2} z\Big|_0^{4-x-y} r\, dr\, d\theta=\int_0^{2\pi}\int_0^{2}(4-x-y)\,r\, dr\, d\theta$$

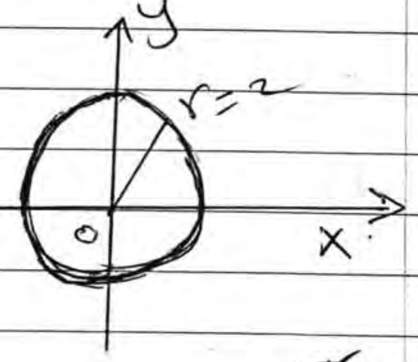
$$=\int_0^{2\pi}\int_0^{2}\left[4r-r^2\cos\theta-r^2\sin\theta\right]dr\, d\theta$$

$$=\int_0^{2\pi}\left[2r^2-\frac{1}{3}r^3\cos\theta-\frac{1}{3}r^3\sin\theta\right]_0^2 d\theta$$

$$=\int_0^{2\pi}\left[8-\frac{8}{3}\cos\theta-\frac{8}{3}\sin\theta\right]d\theta$$

$$=\left[8\theta-\frac{8}{3}\sin\theta+\frac{8}{3}\cos\theta\right]_0^{2\pi}$$

$$=16\pi-0+\frac{8}{3}-\frac{8}{3}=\boxed{16\pi}$$



Chapter

Function and Definite integrals

1. The Error Function
2. The Gamma Function
3. The Beta Function
4. The Factorial function.

1 The Error Function

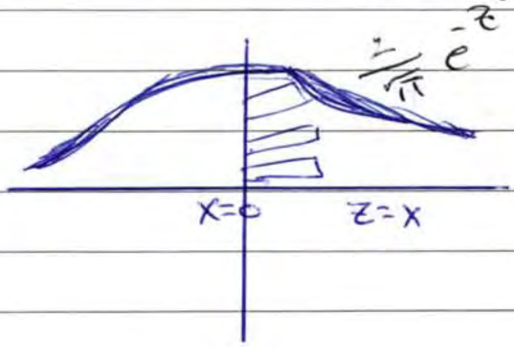
the error function denoted erf, is defined by the integral

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\, dz.$$

and clearly represents the area under the curve e^{-z^2}, from $z=0$ to $z=x$

z is a dummy variable because it only enables the curve to be described and any variable would do this.

the variable z is eliminated by the limits of integration thus leaving x as the only variable.



The factor $\frac{2}{\sqrt{\pi}}$ is introduced for convenience so that

$$\boxed{\operatorname{erf} \infty = 1}$$

$$\operatorname{erf} \infty = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-z^2} dz$$

or $\operatorname{erf} \infty = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} dx.$

To prove that $\operatorname{erf} \infty = 1$

Take $I = \int_0^\infty e^{-x^2} dx$ and solve

$$I = \int_0^\infty e^{-x^2} dx$$

$$I^2 = \left(\int_0^\infty e^{-x^2} dx\right)^2 = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy.$$

$$I^2 = \int_0^\infty dx \int_0^\infty e^{-(x^2+y^2)} dy = \boxed{\iint_{R_{xy}} e^{-(x^2+y^2)} dy}$$

where the region of integration R_{xy} is the whole of the positive quadrant of the xy-plane.

Now transforming I^2 to Polar Coordinates θ & r using the equations

$x = r\cos\theta$, $y = r\sin\theta$

then the element of area $(dx\,dy)$ has to be replaced by $(r\,dr\,d\theta)$, and x^2+y^2 by r^2

$$I^2 = \iint_{R_{r,\theta}} e^{-r^2}\, r\, d\theta\, dr = \int_{\theta=0}^{\theta=\pi/2} d\theta \int_{r=0}^{r=\infty} e^{-r^2}\, r\, dr$$

$$I^2 = \int_0^{\pi/2} \left(-1/2\, e^{-r^2}\right)\Big]_0^{\infty} d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}$$

$$\text{So } I^2 = \frac{\pi}{4} \Rightarrow I = \frac{\sqrt{\pi}}{2} = \int_0^{\infty} e^{-x^2}\, dx$$

$$\therefore \operatorname{erf}\infty = \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-x^2}\, dx = \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 1$$

Properties of the error Function

① Differentiating error function equation gives directly

$$\frac{d}{dx}\operatorname{erf}x = \frac{2}{\sqrt{\pi}} e^{-x^2}$$

using this result to integrate $\operatorname{erf}x$, by parts give

② Integration of $\operatorname{erf}x$

$$\int \operatorname{erf}x\, dx = x.\operatorname{erf}x - \int x \frac{2}{\sqrt{\pi}} e^{-x^2}\, dx + C$$
$$= x.\operatorname{erf}x + \frac{1}{\sqrt{\pi}} e^{-x^2} + C$$

where c is the constant of integration

$\int erf\, x\, dx$, is sometimes tabulated

under the symbol ($i\, erf\, x$) with

$c = -\frac{1}{\sqrt{\pi}}$ so that $i\, erf\, 0 = 0$

Another related function which is sometimes tabulated is the Complementary error function ($erfc\, x$), this is defined by the equation

$$erfc\, x = 1 - erf\, x = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-x^2}\, dx$$

(2) The gamma Function $\Gamma(x)$

This is defined by the integral,

$$\Gamma(x) = \int_0^\infty t^{n-1} e^{-t}\, dt$$

set $x = x+1$, then integrate by Parts gives,

$$\Gamma(x+1) = \int_0^\infty t^{x} e^{-t}\, dt = -t^{x} e^{-t}\Big]_0^\infty + x\int_0^\infty t^{x-1} e^{-t}\, dt$$

$$\Gamma(x+1) = x\,\Gamma(x), \quad (x>0)$$

when $x = n$, and n being a Positive integer >1 we have,

$$\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\,\Gamma(n-1) = \cdots\cdots = n!\,\Gamma(1)$$

and since $\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1$, give,

$$\Gamma(n+1) = n!$$

if $n=1 = \Gamma(2) = 1\,\Gamma(1) = 1!$
if $n=2 = \Gamma(3) = 2\,\Gamma(2) = 2 * 1\,\Gamma(1) = 2!$
if $n=3 = \Gamma(4) = 3\,\Gamma(3) = 3*2\,\Gamma(2) = 3\times 2\times 1\,\Gamma(1) = 3!$

in which $\Gamma(\frac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}} e^{-t}\, dt$

and solved by Putting $t = u^2$

and integrating, $\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t}$

$\because t = u^2 \Rightarrow dt = 2u\,du$ & $e^{-t} = e^{-u^2}$

$t^{-\frac{1}{2}} = (u^2)^{-\frac{1}{2}} = u^{-1} = \frac{1}{u}$

Now substinte for those in main integral you get

$\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-u^2}\, du = 2 \cdot \frac{\sqrt{\pi}}{2} = \boxed{\sqrt{\pi}}$

Since we had Prove $\int_0^\infty e^{-u^2}\, du = \boxed{\frac{\sqrt{\pi}}{2}}$

Previously so $\Gamma(1.5) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$

The relation $\Gamma(x+1) = x\Gamma(x)$ is also useful in defining the Γ - function for negative values of x

if were-writting this relation

$\Gamma(x) = \frac{\Gamma(x+1)}{x}$

- Note that $\Gamma(x)$ becomes in finite at $x = 0$

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Note :- All negative integral values of x becomes infinite, and it is important to emphasise that the value of $\Gamma(x)$ for negative values of x are not given by the integral form but by the recurrence relation

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}$$

For tables of the Γ- function see Jahnke and Emde ("Tables of Functions". Dover Publications, 1945.)

We now show how certain integrals may be evaluated in terms of the Γ-function.

(3) The Beta function (β)

The beta-function $\beta(m,n)$ is defined by the integral:-

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$$

for $m > 0$ & $n > 0$

It is necessarily symmetric in m and n

$$\beta(n,m) = \int_0^1 x^{n-1} (1-x)^{m-1}\, dx$$

Since, by putting $1-x = u$ and $du = -dx$ in above integral will give the formula of beta-function $\beta(n,m)$

$$\beta(n,m) = -\int_1^0 u^{m-1} (1-u)^{n-1}\, du = \beta(m,n).$$

An alternative form of the beta-function obtained by putting

$x = \sin^2\theta$ and $dx = 2\sin\theta \cos\theta\, d\theta$

$1 - x = \cos^2\theta$ and substitute in main eq of $\beta(m,n)$ to get

$$\beta(m,n) = \int_0^{\pi/2} \sin\theta^{2(m-1)} \cos\theta^{2(n-1)} \cdot 2\sin\theta \cos\theta\, d\theta$$

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where the limits of integration changed from $0 \rightarrow 1$, to $0 \rightarrow \frac{\pi}{2}$, by

Putting $x=1$, & $x=0$ in eq. $x = \sin^2\theta$

to find $\theta = 0 \longrightarrow \frac{\pi}{2}$

So $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

From which we may derive a reduction formula relating $B(m,n)$ as

$$B(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!} B(1,1),$$

where $B(1,1) = 1$ and $B(\frac{1}{2},\frac{1}{2}) = \pi$

and how to prove that $B(1,1) = 1$ and $B(\frac{1}{2},\frac{1}{2}) = \pi$ is by using any one of the above two

relations and it leave it for you to prove it. the relation between Gamma and Beta function is

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

(4) The Factorial Function

This is defined by the integral.

$$n! = \int_0^\infty e^{-t} t^n \, dt = \Gamma(n+1)$$

also $n! = n(n-1)(n-2) - - - - \times 3 \times 2 \times 1 \times 0!$

where $0! = 1$ and to Prove that

$$0! = \int_0^\infty e^{-t} \, dt = -e^{t}\Big|_0^\infty = 1$$

For example $3! = 3 \times 2 \times 1 \times 0! = 3 \times 2!$

$4! = 4 \times 3 \times 2 \times 1 \times 0! = 4 \times 3!$

$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 0! = 7 \times 6! = 7 \times 6 \times 5!$

$= 7 \times 6 \times 5 \times 4!$

Class

① $\boxed{\text{Prove that } \Gamma(n+1) = n!}$

This is defined by the integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

Set $x = x+1$, then integrate by parts gives

$$\Gamma(x+1) = \int_0^\infty t^{x} e^{-t}\, dt = -t^{x} e^{-t}\Big]_0^\infty + x\int_0^\infty t^{x-1} e^{-t}\, dt$$

$$\Gamma(x+1) = x\,\Gamma(x) \quad , \quad (x > 0)$$

when $x = n$, and n being a positive integer ≥ 1
we have, $\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n!\,\Gamma(1)$

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\,\Gamma(n-1) = \cdots = n!\,\Gamma(1)$$

and since $\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1$, gives

$$\Gamma(n+1) = n!$$

② Prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Solution:-- $\Gamma(\frac{1}{2}) = \int_0^\infty t^{-1/2} e^{-t} dt$, and solved by

Putting

$t = u^2$ and integrating $\Gamma(\frac{1}{2}) = \int_0^\infty t^{-1/2} e^{-t} dt$

$\therefore t = u^2 \Rightarrow dt = 2u\,du$ and $e^{-t} = e^{-u^2}$

and $t^{-1/2} = (u^2)^{-1/2} = u^{-1} = \frac{1}{u}$

Now substitute for those in main integral you

get $\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-u^2} du = \cancel{2} \cdot \frac{\sqrt{\pi}}{\cancel{2}} = \boxed{\sqrt{\pi}}$