الجامعة التكنولوجية

قسم الهندسة الكيمياوية

المرحلة الرابعة

اختيار الافضل

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CE 134	**Optimization**	Units	2
		Theoretical	2 hr/week
		Tutorial	1 hr/week
		Practical	- hr/week

1. **Introduction to Optimization Methods:**

(4 hrs)

2. **Organization of Optimization problems:**

(4 hrs)

3. **Single Variable:**

Analytical methods, numerical methods, graphical methods, numerical Search, restriction Function, unrestricted Function, Direct search, Dichotomous Search, golden Search, Fibonacci Search

(12 hrs)

4. **Multivariables Optimization Methods:**
   - Necessary Conditions For Extreme Values in graphical Cases.
   - Solution by graphical method.
   - Simplex method.
   - Linear Programming and application in Chemical engineering (transportation mixing)

(10 hrs)

Introduction to Optimization

What is optimization?

* An act, process or methodology of making something (as a design, system or decision) as fully perfect, functional or effective as possible, that is to achieve optimum.
* A collective process of finding the best conditions required to achieve the best results from a given situation.
* Optimization is one of the major quantitative tools in the machinery of decision making. A wide variety of problems in the design, construction, operation and analysis of chemical plants can be resolved by optimization.

Some Other Definitions :-

Optimum value :- It is a technical term including quantitative measurements and mathematical analysis to determine the best setting (maximum or minimum) of a dependent variables.

Optimization Procedure :- The process of determining the maximum or minimum value of some criterion function.

Optimization Problem :- Is the specification of the variables that need to be optimized.

Why are engineers interested in optimization? What benefits result from optimization versus intuitive decision making?

* Engineers work to improve the initial design of process and equipement
* Engineers strive for enhancements in the operation, in order to realize:
  1- largest production,
  2- greatest profit,
  3- minimum cost,
  4- least energy usage.
* In plant operation, benefits arise from improved plant performance such as
  1- Improved yields of valuable products or reduced yields of contaminants.
  2- Reduced energy consumption.
  3- Higher processing rates.
  4- Longer times between shutdowns.
  5- Reduced maintenance costs.
  6- Less equipment wear.
  7- Better staff utilization.

In a typical industrial company there are three levels in which optimization is used:

1- Management
2- Process design and equipment specification.
3- Plant operation.

Process design and equipment specification is usually performed prior to the implementation of the process, and management decisions to implement des are usually made far in advance of the process design step.

Optimum Economic Design: It is based on the best or most favorable conditions that give the least cost per unit of time or maximum profit per unit of production.

Example 1: Determination of pipe diameter to use when pumping a given amount of fluid from one point to another.

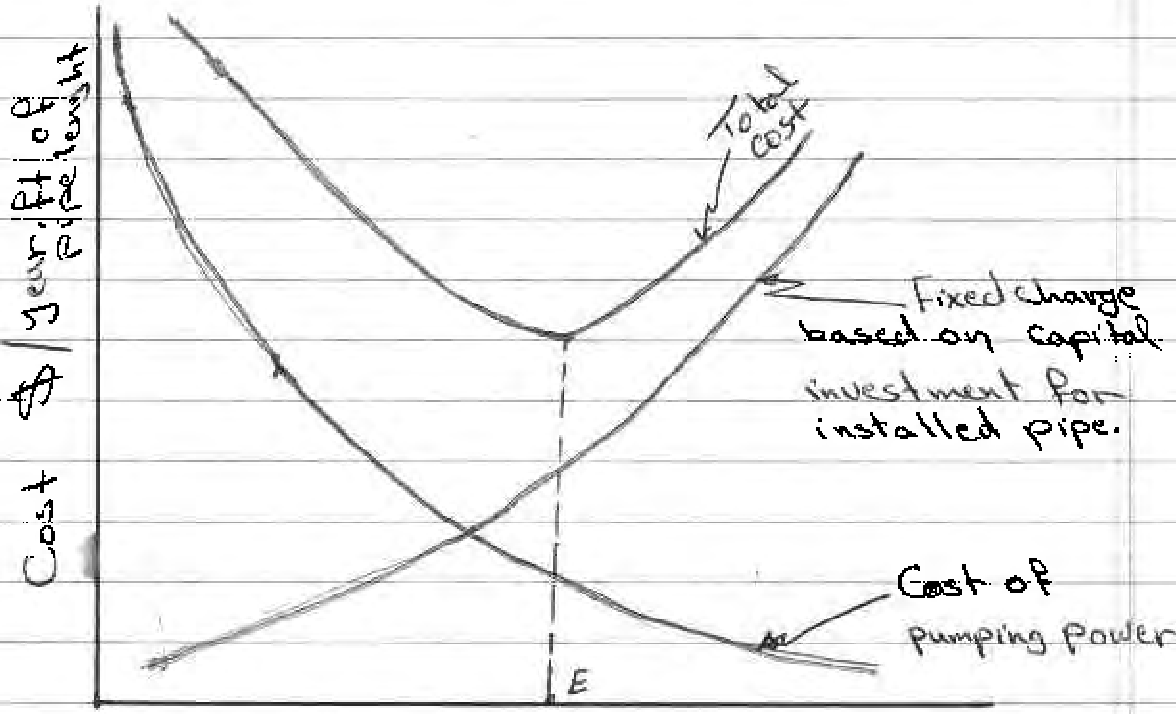


Figure 1: Determination of Optimum Economic Pipe Diamter for Constant Mass Througput Rate.

* Same final results can be accomplished by using infinite number of different pipe diameters.
* economic balance can show that one particular pipe diameter gives the least total cost.
* Total cost = Pumping cost + Fixed cost for the installed piping system

* Pumping Cost increases with decreased size of pipe diameter, why?
* The fixed charges for the pipeline become lower when smaller pipe diameters are used, why?
* The optimum economic diameter is found where the sum of the pumping costs and fixed costs for the pipeline becomes a minimum.
* Eventhough, the engineer must chooses the cheapest design by considering the quality of the product and the operation as well as the total cost.

Example 2:- The design of a distillation column is ordinarily based on specifications giving the degree of separation required for a feed supplied to the unit at a known composition, temperature and flow rate. The chemical engineer must determine the size of the column and reflux ratio necessary to meet the specifications.

* As the reflux ratio is increased, the number of theoretical stages required for the given separation decreases.
* An increase in reflux ratio may result in lower fixed charges for the column and greater costs for the reboiler heat supply and condenser coolant.

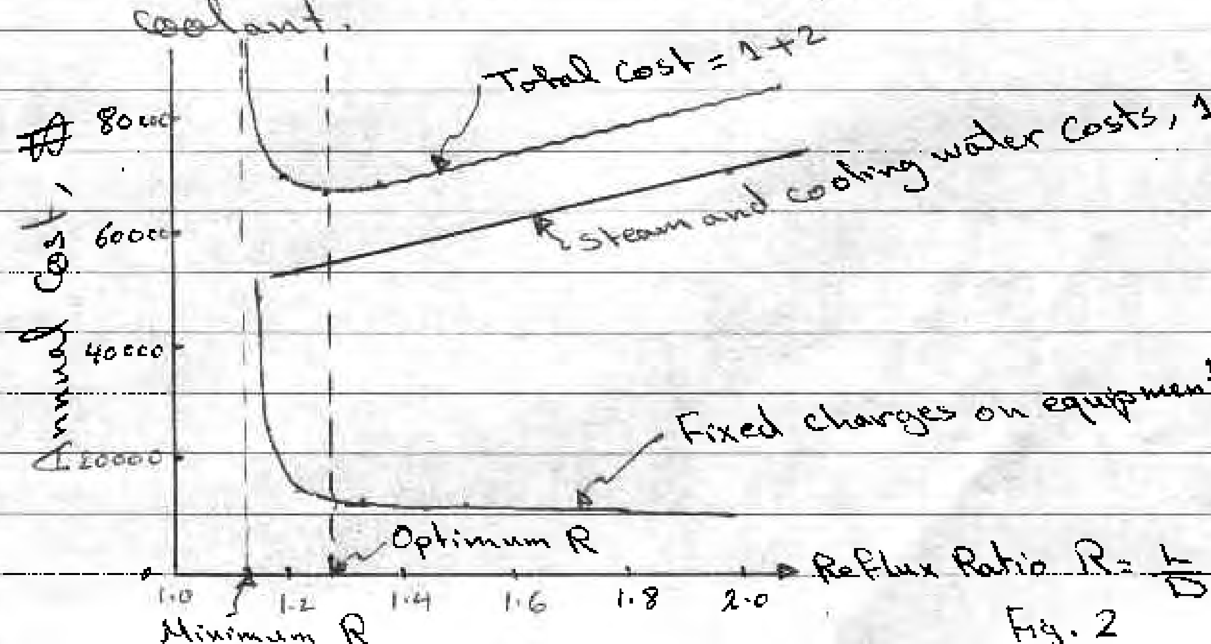


Fig. 2

Optimum Operation Design :-

In every chemical engineering process there are several alternatives which can be used for any given process or operation. For example, formaldehyde can be produced by catalytic dehydrogenation of methanol, by controlled oxidation of natural gas, or by direct reaction between «CO» and «H_2» under special contitions of catalyst, temperature and presste. It is the responsibility of the chemical engineer to choose the best process and to incorporate into his design the equipment and methods which will give the best results. The engineers generally replace the word "best" by "Optimum". Many processes require definite conditions of temperatures, pressure, contact time, catalyst type, composition and many other variables. It is often possible to make a partial separation of these optimum conditions from direct economic considerations. So the Optimum Operation Design is a tool or step in the development of an optimum economic design.

Example :- Determination of the operation conditions for the catalytic oxidation of SO_2 to SO_3

* Suppose all variables, size, flow rate, catalyst, concentration are fixed
* The only possible variable is the operating temperature

* High Temp :-
  Lower yield of SO_3 because of the equilibrium
* Low Temp :-
  Poor yield because of lower rate of reaction of SO_2 with O_2

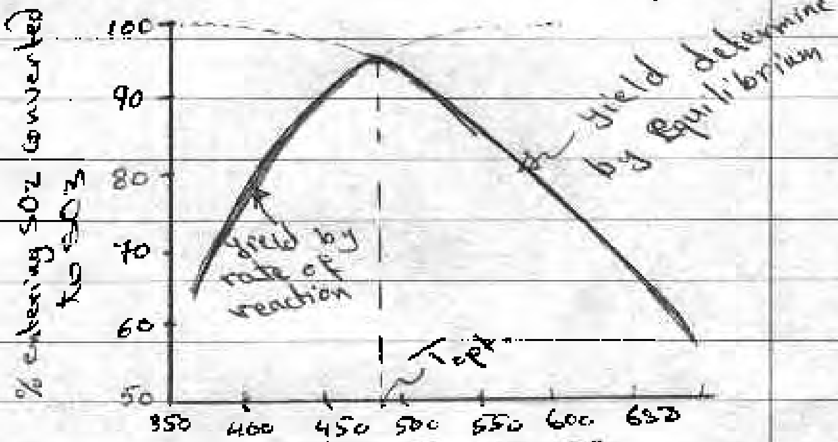


Fig 3 :- Optimum operation temp. for SO_2 to SO_3

Example :- Determination of Optimum Reflux Ratio

A bubble-cap distillation column is being designed to handle 700 lb feed/hr. The unit is to operate at 1 atm. The feed contains 45 mole% benzene and 55 mol% toluene and enters at its boiling point. The overhead product must contain 92 mol% benzene and the bottoms must contain 95 mol% toluene. Determine the following

a) The optimum reflux ratio as moles liquid returned to tower per mole of distillate product withdrawn.

b) The ratio of the Optimum Reflux Ratio / Min. Reflux Ratio.

c) The percent of the total variable cost due to steam consumption at the optimum condition

Data

* C_p for both benzene and toluene = 40 Btu/lbmol.F°
* λ for " " " " = 13700 Btu/lbmol
* U overall H.T.C = 80 Btu/hr.ft².F° in the reboiler
* and = 100 Btu/hr.ft².F° in the condenser
* T_F = 201 °F
* T_D = 179 F°
* T_B = 277 °F
* ΔT driving force in the reflux condenser may be based on an average cooling water temperature of 90 F
* and the change in cooling water temperature is 50 F
* Saturated steam at 60 psia is used in the reboiler where the temperature of the condensing steam is 292.7 °F and the heat of condensation = 915.5 Btu/lb
* Column diameter based on maximum allowable vapor velocity of 2.5 ft/sec at the top of the column.
* Overall Plate Efficiency 70 %
* The unit is to operate 8500 hr per year

Cost Data

* Cost of steam = \$ 0.5 / 1000 lb.
* Cost of cooling water = \$ 0.03 / 1000 gal

or = \$ 0.036 / 10.000 lb.

* The cost of piping, insulation and instrumentation can be estimated to be 60% of the cost for the installed equipment.
* Annual fixed charges amount to 15% of the total cost for installed equipment, piping, insulation & instrumentation.
* The following costs are for the installed equipment, piping and include delivery & erection costs

Bubble-Cap Distillation Column

Values may be interpolated

Diameter, inch	$/plate
60	800
70	1000
80	1230
90	1500
100	1800

Condenser - shell & tube Heat Exchanger

Heat transf area ft^2	$
800	6500
1000	7500
1200	8400
1400	9200
1600	9900

Reboiler - shell & tube Heat Exchanger

Heat transfer area, ft^2	$
1000	11500
1400	14100
1800	16400
2200	18500
2600	20200

Solution :-

The variable costs involved are cost of column, cost of reboiler, cost of condenser, cost of steam and cost of cooling water. Each of these costs is a function of the reflux ratio, and the optimum reflux ratio occurs at the point where the sum of the annual variabl costs is a minimum. The total variable cost will be determined at various reflux ratios and the optimum reflux ratio will be found by the graphical method. The following is a sample of calculation for reflux ratio = 1.5

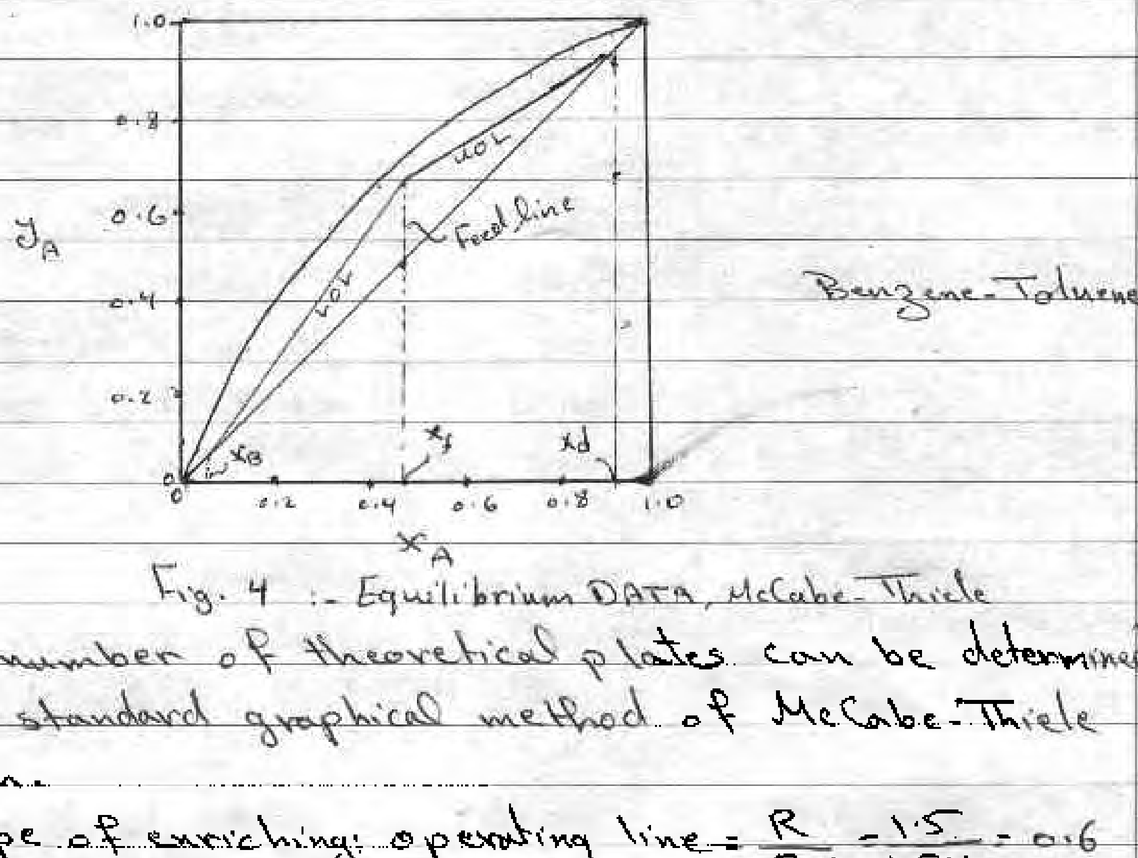


Fig. 4 :- Equilibrium DATA, McCabe-Thiele

The number of theoretical plates can be determined by the standard graphical method of McCabe-Thiele diagram.

slope of enriching operating line $= \frac{R}{R+1} = \frac{1.5}{1.5+1} = 0.6$

theoretical number of stages $= 12.1 - 1 = 11.1$

actual number of stages $= \frac{11.1}{0.7} = 16$

$F x_f = D x_D + B x_B$

$700 * 0.45 = D * 0.92 + (700 - D) * 0.05$

$D = 322$ mole/hr

$\therefore V_n = D(1+R) = 322(1+1.5) = 805$ mole/hr

Vapor velocity at the top of the tower = 2.5 ft/s

From perfect gas law $\frac{V_1/n_1}{V_2/n_2} = \frac{T_1}{T_2}$

$\therefore 2.5 = \dfrac{805 * 359\,(460+179)}{3600\,(460+32) * \frac{\pi d^2}{4}}$

$\therefore d = 7.3$ ft diameter of the distillation column $n = 82.6$

~~Cost of column per plate = \$1430~~

$\therefore$ Annual cost for distillation column $= 1430 * 16\,(1+0.6) * 0.15$

$= \$\,5490$

Annual Cost of Condenser :-

Rate of heat transfer / hr in condenser = mole vapor condensed per hr * molal latent heat of condensation

$\therefore q_c = 805 * 13700 = 11{,}000{,}000$ Btu/hr

since $q_c = UA\Delta T$

$\therefore A = \dfrac{11{,}000{,}000}{100\,(179-90)} = 1240$ ft^2

$\therefore$ Cost per ft^2 $= \dfrac{\$\,8550}{1240}$

$\therefore$ Annual cost for condenser $= \dfrac{\$\,8550}{1240} * 1240\,(1+0.6) * 0.15$

$= \$\,2050$

Annual Cost of Reboiler :-

The rate of heat transfer in the reboiler q_r can be determined by a total energy balance around the distillation column. Base taken at energy level of liquid at 179 °F

Heat Input = Heat Output

$q_r + 700 * 40\,(201-179) = 11{,}000{,}000 + 378 \times 40\,(227-179)$

$\therefore q_r = 11{,}110{,}000$ Btu/hr $= UA\Delta T$

$\therefore A$ = heat transfer area $= \dfrac{11{,}110{,}000}{80\,(292.7-227)} = 2120$ ft^2

$\therefore$ Cost per ft^2 $= \dfrac{\$\,18100}{2120}$

$\therefore$ Annual Cost for reboiler $= \dfrac{\$\,18100}{2120} * 2120\,(1+0.6)\,0.15$

$= \$\,4340$

Annual Cost for Cooling water

$q_c = 11{,}000{,}000$ & the C_p of water $= 1.0$ Btu/lb.F°

$\therefore$ Annual cost of cooling water $= \dfrac{11{,}000{,}000 * 0.036 * 8500}{1.0 * 50 * 10{,}000}$

= \$ 6740

Annual cost of steam

The rate of heat transfer in reboiler = 11,110,000 Btu/hr

$\therefore$ Annual cost for steam = $\dfrac{11,110,000 * 0.5 * 8500}{915.5 * 1000}$

= \$ 51700

$\therefore$ Total Annual cost at reflux ratio 1.5

= 5490 + 2050 + 4340 + 6740 + 51700 = \$ 70320

By repeating the preceding calculation for different reflux ratio, the following table can be prepared

	R	N	d in ft	Ann. Cost in \$					Total cost
				column	Cond.	Reb.	Cooling water	steam	\$
Rmin	1.14	∞	6.7	∞	1870	3960	5780	44300	∞
	1.2	29	6.8	8930	1910	4040	5940	45500	66320
	1.3	21	7.0	6620	1950	4130	6200	47500	66400
	1.4	18	7.1	5920	2000	4240	6470	49600	68230
	1.5	16	7.3	5490	2050	4340	6740	51700	70320
	1.7	14	7.7	5290	2150	4540	7290	55700	7497
	2.0	13	8.0	5210	2280	4800	8100	61800	8219

a) The above data plotted & Ropt can be found at min. total cost per year. From Fig 2 Ropt = 1.25

b) for Rmin slop of U.O.L = $\dfrac{R_{min}}{R_{min}+1}$ = 0.532

$\therefore$ R_{min} = 1.14

$\therefore$ $\dfrac{R_{opt}}{R_{min}} = \dfrac{1.25}{1.14} = 1.1$

c) At the optimum condition

Annual steam cost = \$ 46500

Total Annual cost variable = \$ 66000

$\therefore$ % variable cost due to steam Consumption = $\dfrac{46500}{66000} \times 100$

= 70 %

Examples of Applications of Optimization.

1- Determination of best sites for plant location.
2- Routing of tankers for the distribution of crude and refined products.
3- Pipeline sizing and layout.
4- Equipment and entire plant design.
5- Maintenance and equipment replacement scheduli
6- Operation of equipment, such as tubular reactors, columns, exchangers ... etc.
7- Evaluation of plant data to construct a model of a process.
8- Minimization of inventory charges
9- Allocation of resources or services among several processes
10- Planning and scheduling of construction.
11- Scale formation in evaporators.
12- Insulation thickness.
13- Fitting of mathematical curves to experimenta data to obtain the most accurate representati
14- Fluid dynamics, Mass and Heat transfer applicatio
15- Blending of different types of tobacco an flavor.
16 Road construction between two certain point
17- Quality and number of students for each collage.

And Many Many Other Applications

Optimization Problems involve specifying Four quantities

* Decisions :- Items that need to be figured out to achieve max. efficiency.
* Ranking Function :- A method to rank different choices of decisions.
* Rules and Restrictions :- Specifying limitations on choices of decision values.
* Parameters :- Information necessary in specifying ranking function and rules.

Optimization models are mathematical representation of optimized problem, it is called also mathematical model which analyze or solve the optimization problem.

These models have :-

parameters → represents the given data

Decision variables → represent items that need to be determined.

Constraints → represent limitation on the choice of decision variables, either internal or external imposed by the designer

Objective function → give the ranking of different choices.

General Procedure of Optimization

1- Define a suitable objective for the problem.
2- Examine the restrictions imposed upon the problem.
3- Choose a system or systems to study.
4- Examine the structure of the system and the inter-relationship of the system elements.
5- Construct a mathematical model for the system.
6- Examine and define the internal restrictions planned upon the system variables.
7- Carry-out the simulation by expressing the objective function.
8- Verify the proposed model.
9- Determine the optimum solution and discuss the nature of system conditions.
10- Using the information that obtained, repeat the procedure untill a satisfactory results are found.

Example :- We do complete formulation for a well-known problem called diet problem. Given a set of food (e.g. milk, Chocolate, Raisn-Bran, Pizza) and their nutrient/calorie values, find a diet minimizing the daily cost of food.

* Problem statement :-
  1- Decision :- Quantities of different food items to be consumed daily.
  2- objective :- minimizing the daily cost of food.
  3- Restrictions or constraints :-
     a- total fat content in the diet does not exceed some limit,
     b- total calories do not exceed some limit
     c- total protein intake is at least some minimum amount.
  4- Parameters :- nutrient/calorie, cost

1/ Decision : For each food item, how much of it we should consume daily?

The decision variables can be defined as follow :-

X_m = Gallons of milk consumed daily.
X_c = Bars of chocolate consumed daily.
X_r = Packs of Raisin-Bran " "
X_z = Pizza consumed daily

2. Objective Function:-

Suppose our objective is to minimize our daily cost.

Let C_i for $i = m, c, r, z$ be the current cost per unit of each item i.

So the objective function:-

$$f(x_m, x_c, x_r, x_z) = C_m x_m + C_c x_c + C_r x_r + C_z x_z$$

3. Constraints or restrictions

* Restrictions on fat, calories and protien intake.

Suppose f_i, w_i & P_i, for $i = m, c, r, z$ are the values of fat, calories and protein per unit item respectively.

Then

$$f_m x_m + f_c x_c + f_r x_r + f_z x_z \leq F$$

$$w_m x_m + w_c x_c + w_r x_r + w_z x_z \leq W$$

$$P_m x_m + P_c x_c + P_r x_r + P_z x_z \geq P$$

So the complete model in words is

The Goal → minimum cost such that

1- Fat requirement

2- Calorie requirement

3- Protein requirement

4- Non of the intake amount is negative

i.e

$$x_m \geq 0, \quad x_c \geq 0, \quad x_r \geq 0, \quad x_z \geq 0$$

Formulation of Optimization Model:

1 - Type of Variables: or Decision Variables.
These are defined to capture decisions that need to be made. These variables have different types depending on the values they can take.

The basic variable types are:

a- Continuous: This can take any real value such as, by how much should I change my current investment in stocks?

b- Continuous - non-negative: This can take only non-negative values $x \geq 0$. Such as how much milk should I drink every day

c- Binary: This can take values 0 & 1 only, $x \in \{0, 1\}$. Such as, should a new warehouse be setup in Baghdad?.

d- Integer: This can take any integer value, $x \in Z$, where Z is the set of integer. If it is required to be non-negative then $x \geq 0$. Such as how many workers should be hired to meet lunch time demand in a café e- Finite Sets :- This can take a small set of values $x \in S$ where S is the set of values x can take. Such as which road should I take to collage today.

2- Objective Function :- Can be defined as a function of decision variables whose output is a number. There are uncountable possible functions of this kind, these are classified into two groups :-

a- Linear functions :-

$$f(x_1, x_2, x_3) = x_1 + x_2 + 5x_3$$

b- Non-Linear functions :-

polynomials $\rightarrow f(x,y,z) = x^2 + y^2 + z^2$
crossterms $\rightarrow f(x,y,z) = xy$
Eponentials $\rightarrow f(x) = e^x$
Maximum $\rightarrow f(x,y,z) = \max\{x,y,z\}$
Absolute $\rightarrow f(x) = |x|$

3- Constraints :-

a- Linear constriants which are of three types

$$\geq \quad \leq \quad =$$

b- Non-Linear constraints

$$x^2 + y^2 \leq 1 \quad , \quad 4 < x < 1$$

Optimization Techniques :-

It is the process of determining the maximum or minimum value of some criterion function (objective function).

let us denote the quantity to be optimized or the objective function as

$$J(x)$$

where x may be

1- Single variable (independent).
2- A vector of such variables
3- A function of some other independent variables.
4- Or even a vector of function of of several independent variables

We assume that $J(x)$ is a scalar-valued function of x. If $J(x)$ is a continuous function of a single variable x is restricted to values in the range $a < x < b$. Figure 5 illustrates some possible shapes for various objective functions.

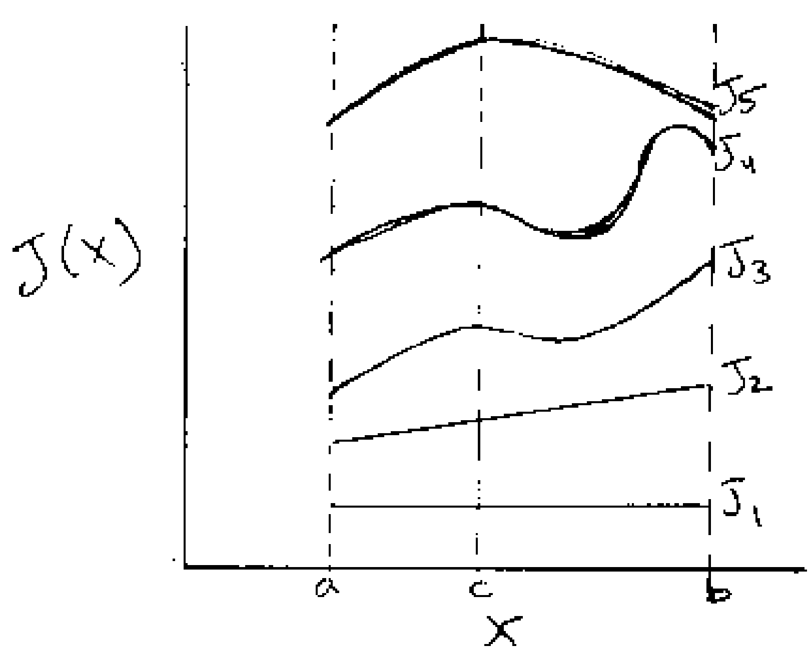


Fig5:- Objective Functions shapes

For J_1 :- There is no unique value for x which gives a maximum value for the objetive function.

For J_2 & J_3 :- Maximum value occurs for x at the boundary value b.

For J_3 & J_4 :- The value $x = c$ gives a locally maximum value for J but not the true maximum.

For J_5 :- The value $x = c$ does give the true maximum.

* If the objective function has only one maximum value in a given range for x it is called Unimodal function. From Fig 5, J_2 and J_5 are unimodal functions.

* If there is more than one maximum or minimum, a local optimum should be found for each then these optimum values should be compared to determine the true optimum

Local and Global Extremes :-

Consider Figure 6, which shows that the objective function $J(x)$ has several peaks within the interval considered, where $a < x < b$. Point A gives maximum value, whereas points B and C give other maximum points for the function $J(x)$

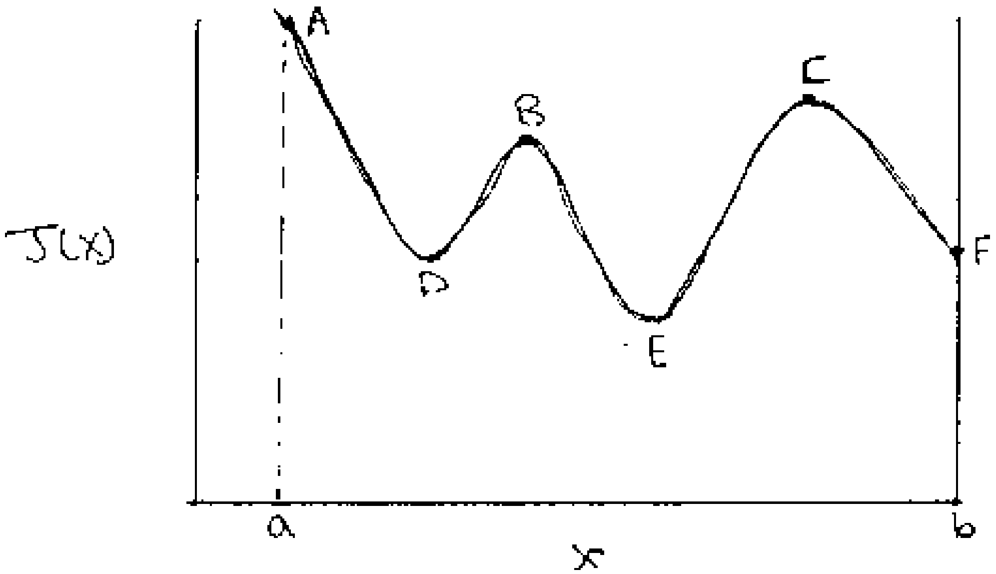


Fig 6

Point A :- Is called Global Maximum
Points B & C :- Are called Local Maximum
Point E :- Is called Global Minimum
Points D & F :- are Local Minimum

From calculus, to find min. or max. values, the first derivative = 0

$$\left.\frac{dJ}{dx}\right|_{x^*} = 0$$ where x^* is the optimum value of x.

$J''(x^*) < 0$ (-ve) local max.

$J''(x^*) > 0$ (+ve) local min.

$J''(x^*) = 0$ Point of inflection or saddle point.

There are two approches to the optimization problems to be solved, either analytical method or numerical method

1- Analytical Method :-

Used for single variable problems (unrestricted).

Rules about extremes (stationary Points) are:-

a) Extremes of $J(X)$ can occur only where

$$\frac{dJ}{dX} = 0$$

b) If at points determined from (a) certain derivative vanish (=0), then the next derivative which does not vanish is examined for sign :-

$$\text{If } \frac{dJ}{dX} = \frac{d_2 J}{dX^2} = \cdots = \frac{d_n J}{dX^n} = 0$$

then

$$\frac{d_{n+1} J}{dX^{n+1}} \text{ is either } (+ve) \text{ or } (-ve).$$

If n is even $\longrightarrow$ point of inflection

if n is odd $\longrightarrow$ n+1 examined $\longrightarrow$ (-ve) max.

$\longrightarrow$ (+ve) min.

c) If $\frac{dJ}{dX}$ changes from +ve to zero to negative (-ve) it gives max.

If $\frac{dJ}{dx}$ changes from -ve $\rightarrow$ zero $\rightarrow$ +ve it gives min.

If sign does not changes, there is no extremes.

Example 1 :-

$$y = a^2 - x^2 \qquad \text{Find the extremes}$$

The necessary conditions for extremes (max or min.) is $y' = 0$

then $y' = -2x = 0$ at $x = 0$

So the extreme at $x = 0$ & $y = a^2$ is max

$y'' = -2$ (-ve) so it is indeed a maximum.

Example 2 :- $y = x^3 - 9x^2 + 24x$ Find the extremes.

$$y' = 3x^2 - 18x + 24 = 0$$
$$(3x - 12)(x - 2) = 0$$
$$\therefore x = 4 \quad , \quad x = 2 \qquad \text{Two extreme points}$$

$$y'' = 6x - 18$$
$$y''(x=2) = -6 \longrightarrow \text{max}$$
$$y''(x=4) = +6 \longrightarrow \text{min}$$

Example 3 :- Assume x is the insulation thickness of a pipe transferring steam between two points. The two costs which has to be considered are

$$\text{Fixed Cost} = ax + b$$
$$\text{Cost of heat losses} = \frac{c}{x} + d$$

Find the optimum insulation thickness to minimize the total cost.

$x \geq 0$, a, b, c & d constants.

Solution :

Total Cost = $C_T = ax + b + \frac{c}{x} + d$

where a, b, c and d are costants

For minimum cost $\frac{dC_T}{dx} = 0$

$\therefore \quad \frac{dC_T}{dx} = a - \frac{c}{x^2} = 0$

$\therefore \quad x = \left(\frac{c}{a}\right)^{1/2}$ This is either the optimum point or a point of inflection.

$\frac{d_2C_T}{dx^2} = \frac{2c}{x^3}$

so if c postive and x must be postive.

Then

$\frac{d_2C_T}{dx^2} > 0 \qquad (+ve) \rightarrow$ min.

Then $\quad x = \left(\frac{c}{a}\right)^{1/2}$ is the value of x occurs at the optimum which gives min. cost.

Example: $y = 7 + 0.5x - 0.5x^2$

$y' = 0.5 - x = 0 \qquad \therefore x = 0.5$

$y'' = 0.5 \qquad +ve \rightarrow$ min

H.W :-

It is necessary to find optimum L/D ratio that minimize the cost. Find this minimum using f_3 from example 3 page 25.

Restricted Function :-

IF there are restrictions on the independent variables, the optimum may not then be at stationary point but lies at the restriction boundary

Eample :- Find the maximum of

$$y = \frac{x^5}{5} - \frac{5x^4}{2} + \frac{35}{3}x^3 - 25x^2 + 24x - 4$$

For a - subjected to the restriction $g = x^2 - (4.1)^2 = 0$

b - " " " $g = x^2 - (4.1)^2 \leq 0$

Solution :

a) $x^2 - (4.1)^2 = 0$ $\therefore x = \mp 4.1$

$y(4.1) = 3.5$

$y(-4.1) = -2264.88$

b) when $-4.1 \leq x \leq 4.1$

Consider the function as unrestricted and choose a stationary point ($y' = 0$), the optimum values at stationary must be compared with objective function at the boundary

$\therefore y' = x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$

$(x-1)(x-2)(x-3)(x-4) = 0$

$x = 1, 2, 3, 4$

$y'' = 4x^3 - 30x^2 + 70x - 50$

$y'' = (x-1)(x-2)(2x-7) + (x-3)(x-4)(2x-3)$

$\therefore$ The results are

x	y	y'	y''	type of pt.
1	4.37	0	-6 ($-$ve)	max
2	3.73	0	2 (+ve)	min
3	4.10	0	-2 ($-$ve)	max
4	3.47	0	6 (+ve)	min

* Two maxs. at $x=1$ & $x=3$. But at $x=1$ gives larger value for y where $y = 4.37$

* y must be tested at the boundary $x = 4.1$
  $y = 3.5$

* So at $x=1$ the global max. occurs which is the required value (at stationary not at the boundary).

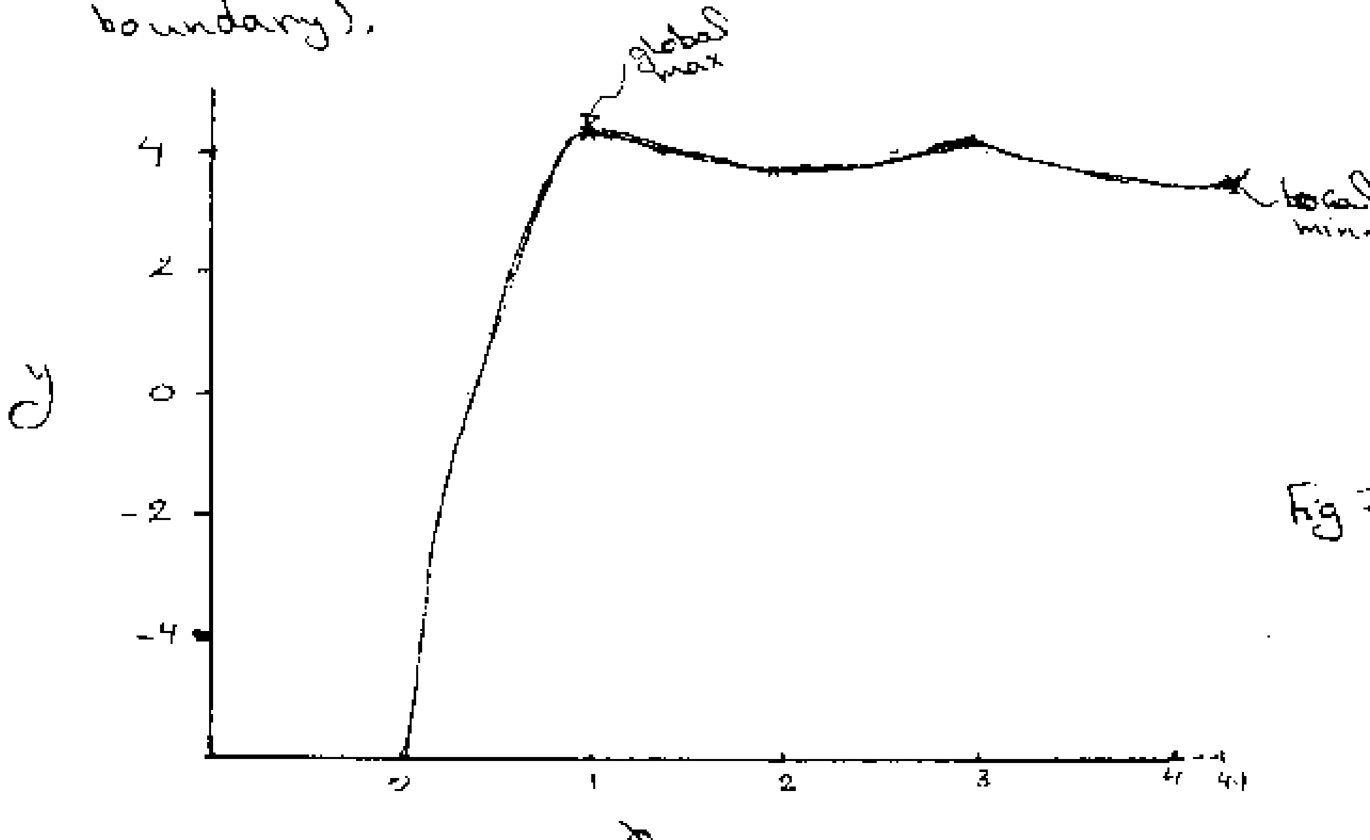


Fig 7

Example:- A plant produces refrigerators at a rate of P units per day. The variable costs per refrigerator have been found to be \$ $(47.73 + 0.1P^{1.2})$. The total daily fixed charges are \$1750, and all other expenses are constant at \$7325 per day. If the selling price per refrigerator is \$173, determine.

a) The daily profit at a production schedule giving the minimum cost per refrigerator.

b) The daily profit at a production schedule giving the maximum daily profit.

c) The production schedule at the break-even point

Solution:-

a) Total cost per refrigerator is

$$C_T = 47.73 + 0.1P^{1.2} + \frac{1750 + 7325}{P}$$

So the decision variable is P (no. of refrigerators/day). and the function of C_T is the objective function.

For minimum cost per refrigerator at production schedule

$$\frac{dC_T}{dP} = 0 = 0.12P_o^{0.2} - \frac{9075}{P_o^2}$$

$\therefore P_o = 165$ units per day for minimum cost per unit.

The daily profit at the production schedule for minimum cost per refrigerator

$$\text{Profit} = \left[173 - \left(47.73 + 0.1P^{1.2} + \frac{9075}{P_o}\right]P_o\right.$$

$$= \left[173 - \left(47.73 + 0.1(165)^{1.2} + \frac{9075}{165}\right]165\right.$$

$= \$ 4040$

b) The Daily Profit at a production schedule (max.)

$$R = \left(173 - 47.73 - 0.1\,P^{1.2} - \frac{1750 + 7325}{P}\right) P$$

For maximum daily profit

$$\frac{dR}{dP} = 0 = 125.27 - 0.22\,P_0^{1.2}$$

$\therefore P_0 = 198$ units per day for maximum daily profit

$$\therefore R_{\substack{max \\ at\,P_0}} = \left(173 - 47.73 - 0.1\,(198)^{1.2} - \frac{9075}{198}\right) 198$$

$= \$ 4400$

c) Total daily profit at break-even point

$$R = 0 = \left(173 - 47.73 - 0.1\,P^{1.2} - \frac{1750 + 7325}{P}\right) P$$

$\therefore P = 88$ units/day at the break-even point

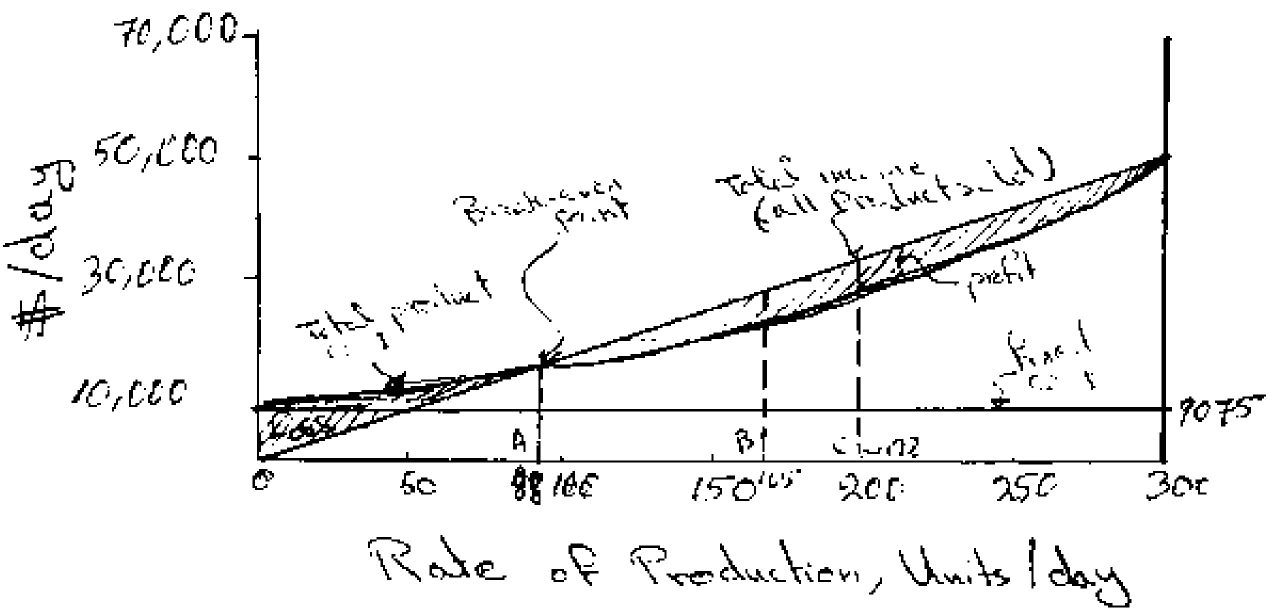


Fig 8

Gradient Discontinuity :-

The function suddenly jumps from one value to another without taking any of the intermediat values, for example

$$y = \frac{1}{x}$$

The function is discontinuous at $x = 0$

$y > 0$ when $x > 0$

$y < 0$ when $x < 0$

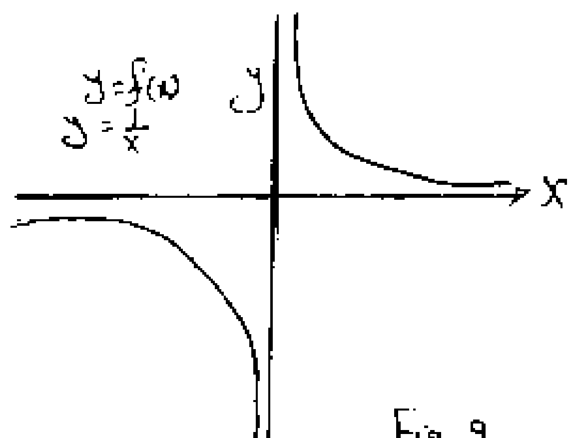


Fig 9

another example

$$y = |x^3|$$

$$y = \begin{cases} x^3 & \text{for } x \geq 0 \\ -x^3 & \text{for } x < 0 \end{cases}$$

$\dot{y} = 3x^2$ at $x = 0$ $\dot{y} = 0$

$y'' = 6x$ at $x = 0$ $y'' = 0$

$y''' = 6$ discontinuous at $x = 0$

The value of $y'''(x)$ at $x = 0$ equal to either 6 or -6 depending upon whether it is approached from a higher or lower value of x so that optimum lying at the discontinuity.

Numerical Methods :.

The analytical method requires the continuity of the function and its derivatives together with a solution for the points at which the first derivative vanishes. Because of these restriction, alternative methods have been suggested, that is some type of search procedure called Numerical Search Method.

Direct Search :-

Assume that for a specified value x_i, we can find the value $f(x_i)$ with known accuracy, and that $f(x_i)$ is unimodal for the range $a < x < b$. Also, assume that we wish to make a fixed number of measurments to determin the optimum of $f(x)$.

for example assume n values of x

$$x_1, x_2, x_3, ---, x_n$$

then find $f(x_1), f(x_2), ---, f(x_n)$

From results $f(x^*)$ will be estimated.

The placing of the trial points x_i is called Search Plan

For the case of two trials x_1 & x_2 where

$$x_1 < x_2$$

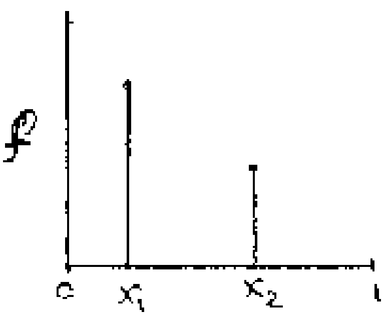
and the function is unimodal at the range $o < x < 1$, there are three possible results (assume search for maxim

a) $f(x_1) > f(x_2) \Rightarrow o \leq x^* \leq x_2$

b) $f(x_1) < f(x_2) \Rightarrow x_1 \leq x^* \leq 1$

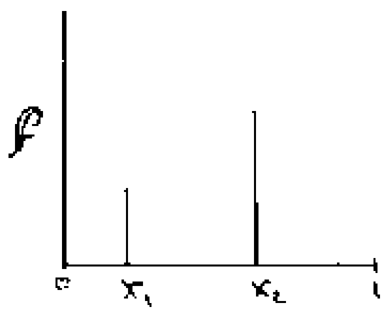
c) $f(x_1) = f(x_2) \Rightarrow x_1 \leq x^* \leq x_2$

As shown in the following figures.



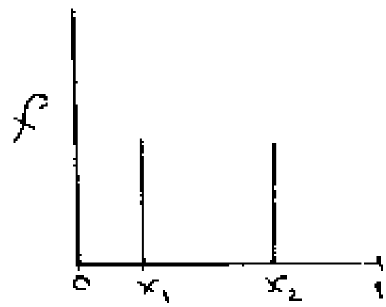
$f(x_1) > f(x_2)$

$0 \leqslant x^* \leqslant x_2$



$f(x_2) > f(x_1)$

$x_1 \leqslant x^* \leqslant 1$



$f(x_1) = f(x_2)$

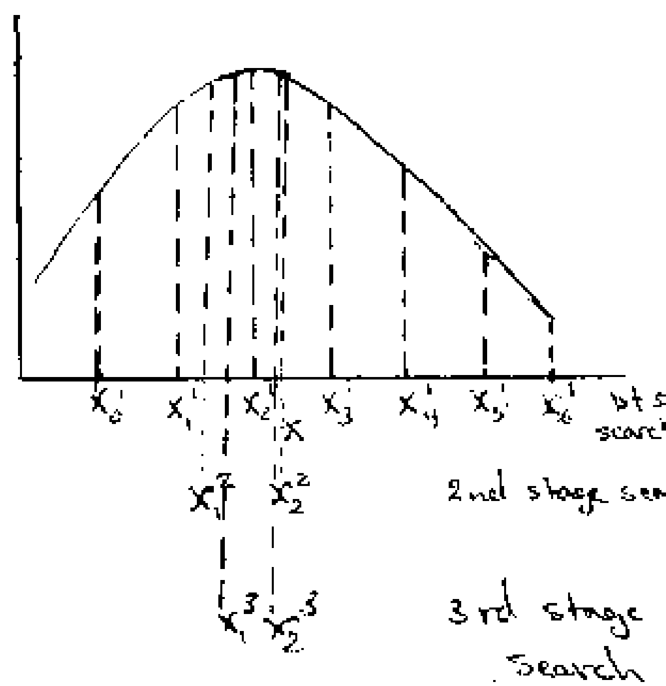
$x_1 \leqslant x^* \leqslant x_2$

Numerical Search – Unrestricted Function :- (open ended)

1. Fixed Step Size :- Unimodal Function

This method is based on the assumptions:-

* Starting from a base point x_0. (or end point in previous search or arbitrary defined) with fixed step size S



$\therefore \quad x_1^1 = x_0^1 + S$

or $\quad x_n^1 = x_{n-1}^1 + S$

For maximum

If $\quad f(x_1') > f(x_0')$

then $\quad x^* \geqslant x_1'$

If $\quad f(x_1') < f(x_0')$

then $\quad x^* < x_1'$

* Search continued using a new base point until the final point x_4' which shows a decrease (1st stage search terminated at x_4')
* Search continued with a new base pt. x_1^2 for higher accuracy

Example:- Determine the minimum of the function

$$y = (x - 100)^2$$

Solution:

1- Choose a base point $x_0^1 = 0$

2- Evaluate $y(x_0^1) = (0 - 100)^2 = 10,000$

3- Choose a step size $S = -3$

4- Evaluate $x_1^1 = x_0^1 + S = -3$

5- Evaluate $y(x_1^1) = (-3 - 100)^2 = 10,609$

Since $y(x_1^1) > y(x_0^1)$

minimum can not be found in the region of $x = -3$

6- take $S = -S = 3$

$\therefore x_2^1 = 3$ & $y(x_2^1) = 9406$

7- First Stage Search

x_i^1	0	-3	3	6	9	-----	93	96	99	102
$y(x_i^1)$	10000	10609	9409	8836	8281	-----	49	16	1	4

8- Second Stage Search: For higher accuracy $S = 0.3$ and $x_1^2 = 99$

x_i^2	99	99.3	99.6	99.9	100.2	100.5
$y(x_i^2)$	1	0.49	0.16	0.1	0.04	0.25

9- Further reduction can be used by making 3rd stage search between 100.2 & 100.3 taking $S = 0.01$ to get highly accurate result.

2- Direct Search with Accelration (DSC) :-

The previous example repeated with acceleration (doubled step size) as follow:-

Step No.	S	Xi		y(xi)	
0	0	0		10000	
1	3	3		9409	
2	6	9		8281	
3	12	21		6241	
4	24	45	x_{i-2}	3025	y_{i-2}
5	48	93	x_{i-1}	49	y_{i-1}
6	96	189	x_i	7920	y_i

The number of measurements required is reduced compared with previous method.

The optimum value X^* will be :-

$$X^* = \frac{1}{2}(X_{i-2} + X_{i-1}) - \frac{\beta_1}{2\beta_{11}}$$

$$\beta_1 = \frac{y_{i-1} - y_{i-2}}{X_{i-1} - X_{i-2}} \quad , \quad \beta_2 = \frac{y_i - y_{i-2}}{X_i - X_{i-2}}$$

$$\beta_{11} = \frac{\beta_2 - \beta_1}{X_i - X_{i-1}}$$

Apply these for the data obtained gives

$$\beta_1 = \frac{49 - 3025}{93 - 45} = -62$$

$$\beta_2 = \frac{7921 - 3025}{189 - 45} = 34$$

$$\beta_{11} = \frac{34-(-62)}{189-93} = 1$$

$$\therefore x^* = \frac{1}{2}(45+93) - \frac{(-62)}{2*1} = 100$$

Example :- Find the maximum of $y = 5 + 3x - x^3$ starting at $x_0 = 0$ using DSC for $S = 0.1$.

solution:

Step No.	S	x_i	$y(x_i)$
0	0	0	5
1	0.1	0.1	5.299
2	0.2	0.3	5.873
3	0.4	0.7	6.757
4	0.8	1.5	6.125
5	1.6	3.1	-15.491
6	-0.4	1.1	6.969

If $x_i = 1.1$, $y_i = 6.969$

$\beta_1 = -0.79$

$\beta_{11} = -3.3$

$x^* = 0.98$

If $x_i = 3.1$, $y_i = -15.491$

$\beta_1 = -0.79$

$\beta_{11} = -5.3$

$x^* = 1.02547$

So we can choose any of these two values.

Restricted Functions :-

a- In the direct search techniques, the variable x is subjected to certain restrictions, ie

$$a \leq x \leq b$$

The search region is divided equally into a certain No. of intervals

b- Allow a set of experiments or measurements to be performed and evaluate the objective function.

1- Uniform Search Method (Preplanned Experiments):-
(Simultaneous search plan)

Let $L_o = b - a$

and L_o is divided into $N+1$ equal intervals, where N is the number of experiments without evaluating $y(a)$ or $y(b)$, then

$$x_i = a + \frac{i\,(b-a)}{N+1} \quad \text{where } i = 1, 2, \cdots, N$$

The optimum will be located within L_N

$$L_N = \frac{2L_o}{N+1}$$

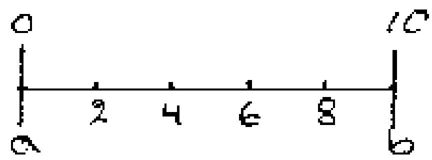
The accuracy α will be

$$\alpha = \frac{L_N}{L_o} = \frac{2L_o}{L_o(N+1)}$$

$$\alpha = \frac{2}{N+1}$$

$$\therefore \quad N \geq \frac{2}{\alpha} - 1$$

For example :- let $0 \le x \le 10$, take $N = 4$, find the optimum max.



1st case:- if $y_4 > y_2$

& $y_4 > y_6$

then the optimum lies in the range $2 < x < 6$

$L_N = 4 = \frac{2 * 10}{4+1}$

2nd case : if $y_6 > y_8$

& $y_6 > y_4$

then the optimum lies in the range $4 < x < 8$

& $L_N = 4$ also

hence the accuracy $\alpha = \frac{L_N}{L_0} = \frac{4}{10} = 0.4$

2 Sequential Uniform Methods

1) Two Experiments :-

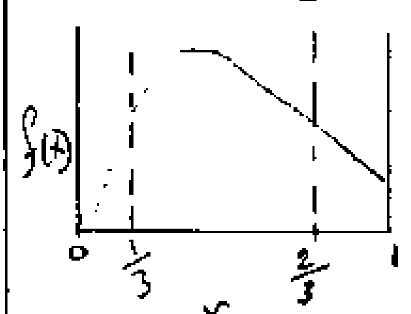
placing two experiments in each cycle, $f(x)$ must be evaluated at $\frac{1}{3}$ and $\frac{2}{3}$ of the search interval. So for two experiments x_1 & x_2 placed within ($L_0 = b - a$), then

$$x_1 = a + \frac{1}{3} L_0$$

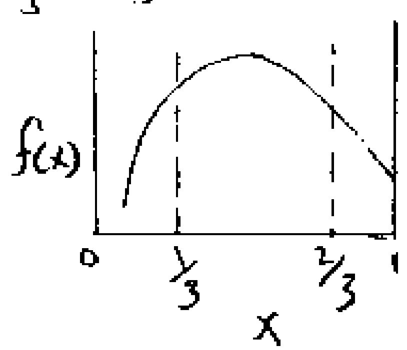
$$x_2 = a + \frac{2}{3} L_0$$

These are from the equation $x_i = a + \frac{i(b-a)}{N+1}$

N = Number of experiments



$0 < x < \frac{2}{3}$



$\frac{1}{3} < x < 1$

For N = number of experiments
and m = number of cycles

$$L_N = L_{2m} = \left(\frac{2}{3}\right)^m L_0$$

i.e.
For the 1st cycle and 2 experiments

$$L_2 = \frac{2}{3} L_0$$

and for the 2nd cycle i.e 4 experiments

$$L_4 = \left(\frac{2}{3}\right)^2 L_0$$

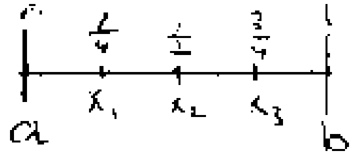
Since $\alpha = \dfrac{L_N}{L_0}$ then $\alpha = \dfrac{\left(\frac{2}{3}\right)^m L_0}{L_0} = \left(\frac{2}{3}\right)^{N/2}$

If N is even $\quad N = 2m \geq 2 \dfrac{\log \alpha}{\log \frac{2}{3}}$

If N is odd $\quad N = 1 + 2m \geq 1 + 2 \dfrac{\log \alpha}{\log \frac{2}{3}}$

b) Three experiments

* $f(x)$ is evaluated at $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ of L_0
* The search interval
  for $0 < x < 1$ will be



$0 - \frac{1}{2}$
$\frac{1}{4} - \frac{3}{4}$
$\frac{1}{2} - 1$

For $N = 3$

1st cycle $\quad L_3 = \frac{1}{2} L_0$
2nd cycle $\quad L_5 = \left(\frac{1}{2}\right)^2 L_0 \quad$ where $L_5 = L_{1+2m}$
mth cycle $\quad L_N = L_{1+2m} = \left(\frac{1}{2}\right)^m L_0 = \left(\frac{1}{2}\right)^{\frac{1}{2}(N-1)} L_0$

Example: Find the optimum minimum point of $y = x^2 - 6x + 2$ in the interval $0 \leq x \leq 10$ using sequential search method with two experiments. The accuracy $\alpha = 0.06$

Solution:

$$N = 2m \geq \frac{2 \log \alpha}{\log 2/3} \geq \frac{2 \log 0.06}{\log 0.6667}$$

$$\therefore N \geq 13.8 \approx 14 \text{ experiments}$$

No. of cycles $m = \frac{N}{2} = \frac{14}{2} = 7$ Cycles

1st Cycle

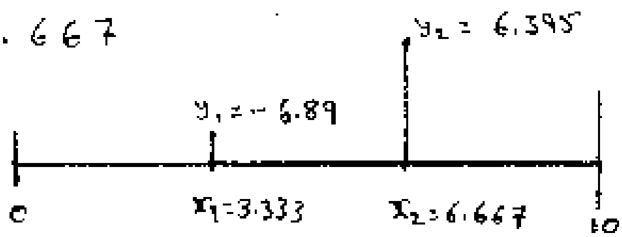
$x_1 = a + \frac{1}{3} L_o = 3.333$ since $L_o = 10$

$x_2 = a + \frac{2}{3} L_o = 6.667$

From the function

$y_1 = -6.89$

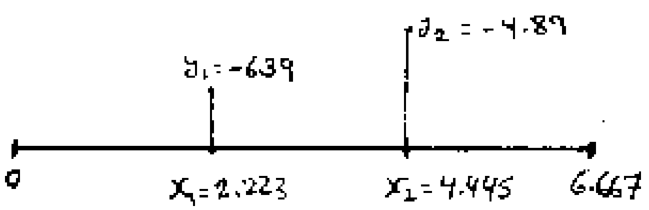
$y_2 = 6.395$



2nd Cycle $0 \leq x \leq 6.667$

$x_1 = 2.223$, $y_1 = -6.39$

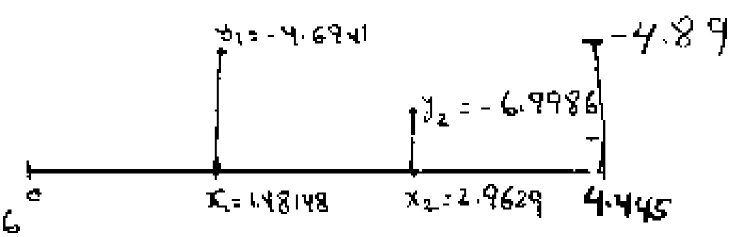
$x_2 = 4.445$, $y_2 = -4.89$



3rd Cycle $0 \leq x \leq 4.445$

$x_1 = 1.48148$, $y_1 = -4.6941$

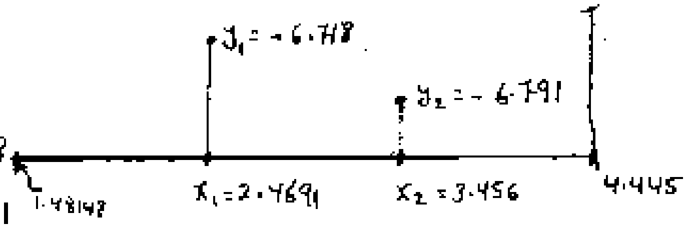
$x_2 = 2.9629$, $y_2 = -6.9986$



4th Cycle $1.48148 \leq x \leq 4.445$

$x_1 = 2.4691$, $y_1 = -6.718$

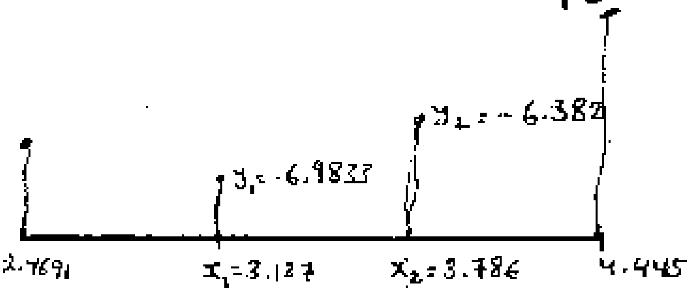
$x_2 = 3.456$, $y_2 = -6.791$



5th Cycle $2.4691 \leq x \leq 4.445$

$x_1 = 3.128$, $y_1 = -6.983$

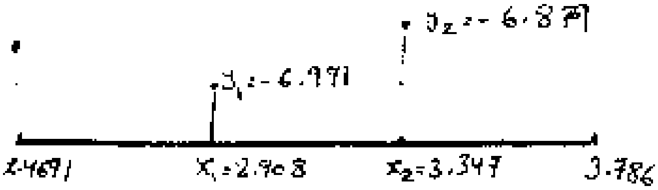
$x_2 = 3.786$, $y_2 = -6.382$



6th Cycle $2.4691 \leq x \leq 3.786$

$x_1 = 2.908$, $y_1 = -6.991$

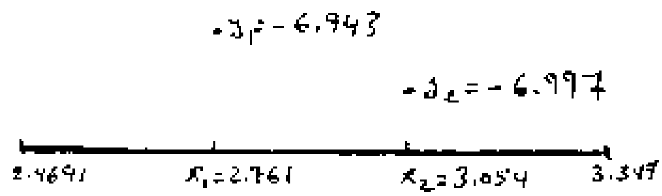
$x_2 = 3.347$, $y_2 = -6.879$



7th Cycle $2.4691 \leq x \leq 3.347$

$x_1 = 2.761$, $y_1 = -6.943$

$x_2 = 3.054$, $y_2 = -6.997$



$\therefore x^* = 3.054$ which is the optimum to give minimum $y = -6.997$

$\therefore L_N = 3.347 - 2.761 = 0.5852$

$$\alpha = \frac{0.5852}{10} = 0.05852$$

H.W ..

1/ Find the maximum optimum of $y = x \sin x$ with search interval $L_o = 0 \longrightarrow 10$ when x in radian using three equally spaced experiments with $\alpha = 0.05$ ..

ans. $x^* = 7.9687$

2/ Solve the previous example (page 38) using

a) Uniform search method (Preplanned experiments).

b) then with 4 experiments.

ans. $x_a^* = 2.9411$

Example :-

Find the optimum minimum point of

$$y = x^2 - 6x + 2 \qquad 0 \leq x \leq 10$$

using sequential method with three experiments use $\alpha = 0.06$.

Solution :-

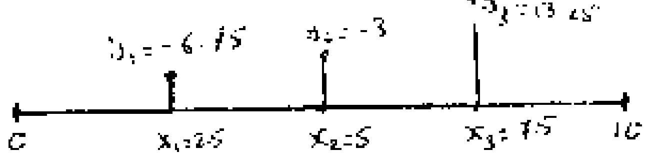
$0 \leq x \leq 10$ take x at $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ of L_o

1st Cycle :-

$x_1 = 2.5$ $\quad y_1 = -6.75$

$x_2 = 5$ $\quad y_2 = -3$

$x_3 = 7.5$ $\quad y_3 = 13.25$

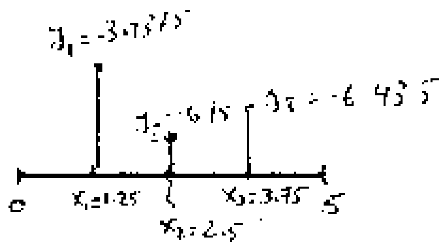


2nd Cycle $\quad 0 \leq x \leq 5$

$x_1 = 1.25$ $\quad y_1 = -3.9375$

$x_2 = 2.5$ $\quad y_2 = -6.75$

$x_3 = 3.75$ $\quad y_3 = -6.4375$

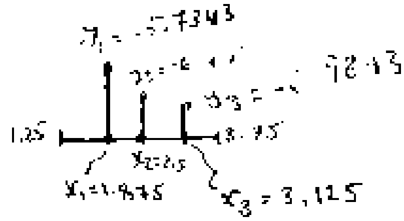


3rd Cycle $\quad 1.25 \leq x \leq 3.75$

$x_1 = 1.875$ $\quad y_1 = -5.7343$

$x_2 = 2.5$ $\quad y_2 = -6.75$

$x_3 = 3.125$ $\quad y_3 = -6.9843$

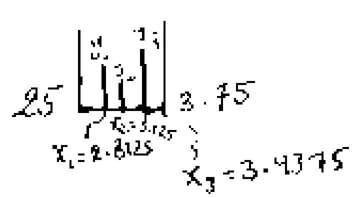


4th Cycle $\quad 2.5 \leq x \leq 3.75$

$x_1 = 2.8125$ $\quad y_1 = -6.9648$

$x_2 = 3.125$ $\quad y_2 = -6.9848$

$x_3 = 3.4375$ $\quad y_3 = -6.8085$



For the 4th Cycle $L_N = 3.125 - 2.5 = 0.625$

$$\therefore \alpha = \frac{0.625}{10} = \frac{L_N}{L_0} = 0.0625 > 0.06$$

so we need another cycle to get an accuracy not more than 0.06

5th Cycle $2.8125 \leq x \leq 3.4375$

$x_1 = 2.9687$ $\quad y_1 = -6.999$

$x_2 = 3.125$ $\quad y_2 = -6.989$

$x_3 = 3.281$ $\quad y_3 = -69208$

So $L_N = 3.125 - 2.8125 = 0.3125$

$$\alpha = \frac{L_N}{L_0} = \frac{0.3125}{10} = 0.03 < 0.06$$

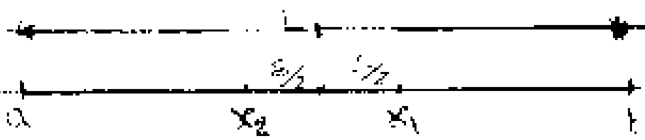
$$\therefore x^* = 2.9687 \quad \& \quad y^*_{min} = -6.999$$

C) Dichotomous Search Plan

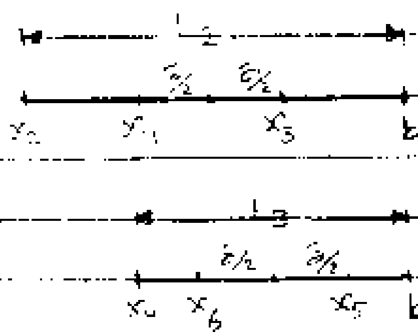
In interval of known limits, the assumption of unimodality is exploited to reduce the interval of uncertainty about the optimum as quickly as possible. For two points placed in a known interval, the interval of uncertainty is minimized for two points placed symmetrically about the midpoint. This is the basis of a dichotomous search. The distance between the two points is called the "Resolution Distance" and is denoted by δ (Delta). The optimal placement of points is given by

For $a \le x \le b$

$$x_1 = b - \frac{b-a+\delta}{2}$$



$$x_2 = a + \frac{b-a+\delta}{2}$$



The search proceeds with the placement of symmetrical points within the remaining interval of uncertainty. For example if the search is for maximum and $y(x_1) > y(x_2)$, the interval of uncertainty is now $x_2 \le x \le b$ and the two new measurements are placed at

$$x_3 = x_2 + \frac{b-x_2+\delta}{2}$$

$$x_4 = b - \frac{b-x_2+\delta}{2}$$

Since δ limits the resolution of measurements, the interval of uncertainty must be greater than δ.

and $x_1 = b - \frac{b-a+\delta}{2}$

$$x_2 = a + \frac{b-a+\delta}{2}$$

The number of experiments is given by

$$(2^K+1)\delta \leq (b-a)$$

Where K is the number of pairs of search points.

Example :-

Determine the optimum maximum of

$$y = x e^{-x} \qquad 0 \leq x \leq 2$$

by dichotomous search , assume $\delta = 0.05$

Solution :-

The No. of experiments is

$$(2^K+1) * 0.05 \leq 2$$

$\therefore K = 5$ which is the greatest value of K satisfying this equation

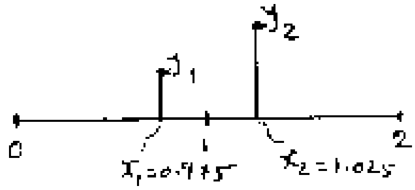
① First pair

$$x_1 = 2 - \frac{2-0+0.05}{2} = 0.975$$

$$x_2 = 0 + \frac{2-0+0.05}{2} = 1.025$$

$y_1 = 0.367763$

$y_2 = 0.367766$



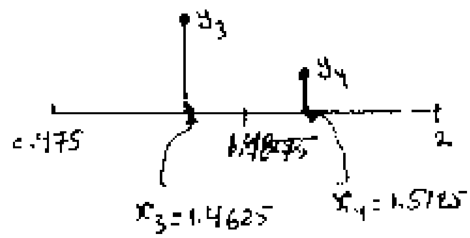
$\therefore\ y_2 > y_1$, so the new interval $0.975 \leq x \leq 2$

(2) 2nd Pair

$x_3 = 2 - \frac{2-0.975+0.05}{2} = 1.4625$, $y_3 = 0.338797$

$x_4 = 0.975 + \frac{2-0.975+0.05}{2} = 1.5125$, $y_4 = 0.333292$

$\therefore\ y_3 > y_4$



The new interval $0.975 \leq x \leq 1.5125$,

Continue to obtain the following table.

i	x_i	y_i	a	b
1	0.975	0.367763	0	2
2	1.025	0.367766		
3	1.4625	0.338797	0.975	2
4	1.5125	0.333292		
5	1.21875	0.360267	0.975	1.5125
6	1.26875	0.35675		
7	1.096875	0.366261	0.975	1.26875
8	1.146875	0.364279		
9	1.035938	0.36747	0.975	1.146875
10	1.085938	0.366396		

* The interval of uncertainty (b-a) must be greater than δ and from the previous table b-a = 0.171875 which is exceeded 2δ. So the best estimate of optimum x can be determined as follow

$$x_1 = \frac{1}{2}(a+b) = \frac{1}{2}(0.975 + 1.085938)$$

$$x_1 = 1.030469$$

and $y_1 = 0.367712$

but from the table the value of y at x = 1.025 gives higher value

$$x_2 = 1.025 \qquad y_2 = 0.367766$$

$$\therefore X^* = 1.025$$

If we use analytical method:

$$y = x e^{-x}$$

$$y' = e^{-x} - x e^{-x} = e^{-x}(1-x)$$

at $y' = 0$ $\quad x^* = 1 \quad$ & $y^* = 0.367879$

$$y'' = -e^{-x} - [e^{-x} - x e^{-x}]$$

$$y'' = e^{-x}(x-2)$$

For $x = 1$

$$y'' = 0.368(1-2) = -ve \quad \therefore \max.$$