Department of Chemical Engineering

2^{nd} class

Mathematic II

By

Dr. Zainab Yousif

Chain Rule

Function of one variable

if $w = f(x)$ *and* $x = g(t)$ *then we can say* $w = f(g(t))$

and we can find $\frac{dw}{dt}$ *by chain ule* $\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}$

Function of two variables

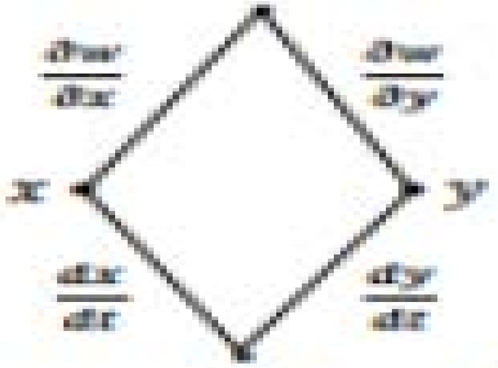
if $w = f(x, y)$ *has continuous partial derivatives* f_x *and* f_y *and if* $x = x(t), y = y(t)$ *are differentiable functions of* t, *then the composite* $w = f\left(x(t), y(t)\right)$ *is a differentiable function of* t *and*

$$\frac{df}{dt} = f_x\left(x(t), y(t)\right) \cdot x'(t) + f_y\left(x(t), y(t)\right) \cdot y'(t).$$

or

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

$w = f(x, y)$ *Dependent variable*



Independent var.

Independent variable

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

Tree diagram

Example 1: Applying the chain Rule

Use the chain Rule to find the derivative of $w = xy$ *with respect to t along the path* $x = \cos t$, $y = \sin t$. *What is the derivative's value at* $t = \frac{\pi}{2}$?

solution: We apply the chain rule to find dw/dt *as follows:*

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$
$$= \frac{\partial(xy)}{\partial x}\cdot\frac{d}{dt}(\cos t) + \frac{\partial(xy)}{\partial y}\cdot\frac{d}{dt}(\sin t)$$
$$= (y)(-\sin t) + (x)(\cos t)$$
$$= (\sin t)(-\sin t) + (\cos t)(\cos t)$$
$$= -\sin^2 t + \cos^2 t = \cos 2t\,.$$

In this example, we can check the result with a more direct calculation. As a function of t,

$$w = xy = \cos t \sin t = \frac{1}{2}\sin 2t,$$

so

$$\frac{dw}{dt} = \frac{d}{dt}\left(\frac{1}{2}\sin 2t\right) = \frac{1}{2}\cdot 2\cos 2t = \cos 2t.$$

In either case, at the given value of t,

$$\left(\frac{dw}{dt}\right)_{t=\pi/2} = \cos\left(2\cdot\frac{\pi}{2}\right) = \cos\pi = -1.$$

Function defined on Surfaces

If we are interested in the temperature $w = f(x,y,z)$ *at points* (x,y,z) *on a globe in space, we might prefer to think of* $x, y,$ *and* z *as function of the variables* r *and* s *that give the points' longitude and latitudes. If* $x = g(r,s), y = h(r,s),$ *and* $z = k(r,s)$, *we could then express the temperature as a function of* r *and* s *with the composite function* $w = f\big(g(r,s), h(r,s), k(r,s)\big)$.

Under the right conditions, w *would have partial derivatives with respect to both* r *and* s *that could be calculated in the following way.*

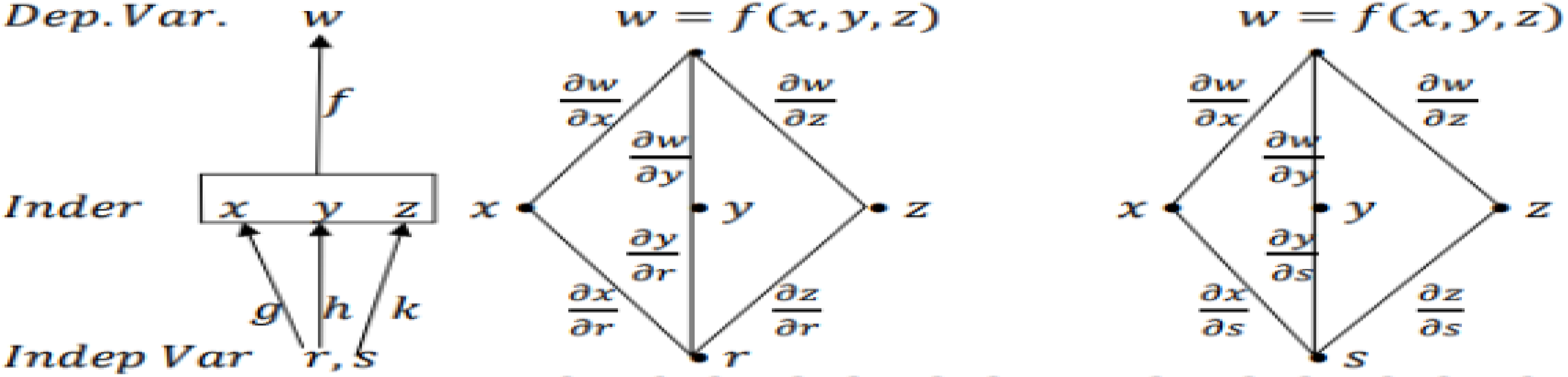
Chain Rule for Two Independent Variables and Three Intermediate Variables

Suppose that $w = f(x,y,z), x = g(r,s), y = h(r,s),$ *and* $z = k(r,s)$.
If all foyr functions are differentiable, then w *has partial derivatives with respect to* r *and* s*, given by the functions*

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

The first of these equations can be derived by holding s *fixed and treating* r *as* t*. The second can be drived in the same way, holding* r *fixed and treating* s *as* t*. The tree diagram for both equations are showen below*



$w = f\big(g(r,s), h(r,s), k(r,s)\big)$. $\quad \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$ $\quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$

$$\frac{\quad}{\partial r} = \frac{\quad}{dx}\frac{\quad}{\partial r} \qquad \frac{\quad}{\partial s} = \frac{\quad}{dx}\frac{\quad}{\partial s}$$

A Formula for Implicit Differentiation

Suppose that $F(x,y)$ is differentiable and that the equation $F(x,y) = 0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$.

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Example 5: Implicit Differentiation

find $\frac{dy}{dx}$ *if* $y^2 - x^2 - \sin xy = 0$

Solution: Take $F(x,y) = y^2 - x^2 - \sin xy$ *, then*

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-2x - y\cos xy}{2y - x\cos xy} = \frac{2x + y\cos xy}{2y - x\cos xy}.$$

Functions of Many Variables

Suppose that $w = f(x, y, z, \dots\dots\dots u)$ is a differentiable function of the variables $x, y, \dots u$ (a finite set) and $x, y, \dots u$ are differentiable functions of $p, q, \dots t$ (another finite sets). Then w is a differentiable function of the variables p through t and the partial derivatives of w with respect to these variables are given by equations of the form

$$\frac{\partial y}{\partial x} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p} + \cdots + \frac{\partial w}{\partial u}\frac{\partial u}{\partial p}.$$

The other equations obtained by replacing p by $q, \dots, t$, one at time. One way to remember this eqution is to think of the right − hand side as the dot product of two vectors with components

$$\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \dots, \frac{\partial w}{\partial u}\right) \quad and \quad \left(\frac{\partial x}{\partial p}, \frac{\partial y}{\partial p}, \dots, \frac{\partial u}{\partial p}\right).$$

Derivatives of w with respect to the intermediate variable	Dervatives of the intermediate variable with respect to the selected independent variable

Extreme Values and Saddle Points
Derivatives Tests For Local Extreme Values

DEFINITIONS (Local Maximum, Local Mini

Let $f(x,y)$ be defined on a region R containing the point (a,b). Then
1 − $f(a,b)$ is a local maximum value of f if $f(a,b) \geq f(x,y)$ for all domain points (x,y) in an open disk centered at (a,b).
2 − $f(a,b)$ is a local minimum value of f if $f(a,b) \leq f(x,y)$ for all domain points (x,y) in an open disk centered at (a,b).

THEOREM: First Derivative Test for Local Extreme Values

If $f(x,y)$ has a local maximum or minimum value at an interior point (a,b) of its domain and if the first partial derivatives exist there, then $f_x(a,b)=0$, and $f_y(a,b)=0$.

DEFINITION Critical Point

An interior point of the domain of a function $f(x,y)$ where both f_x and f_y are zero or where one or both f_x and f_y donot exist is a critical point of f.

DEFINITION Saddle Point

A differentiable function $f(x,y)$ has a saddle point at a critical point (a,b) if in every open disk centered at (a,b) there are domain points (x,y) where $f(x,y) > f(a,b)$ and domain points (x,y) where $f(x,y) < f(a,b)$. The corresponding point $(a,b,f(a,b))$ on the surface $z=f(x,y)$ is called a saddle point of the surface.

Example 1: Finding local Extreme values

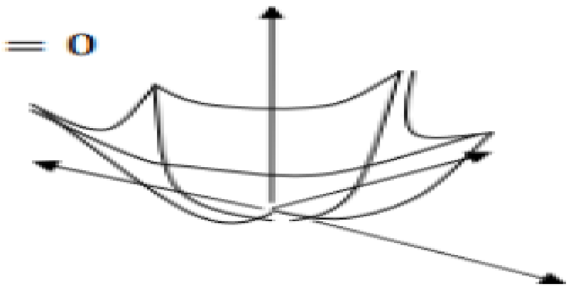
Find the local extreme values of $f(x,y)=x^2+y^2$

Solution: $f_x=2x=0$ and $f_y=2y=0$, $\therefore x=0,\ y=0$

$(a,b)=(0,0)$ so $f(a,b)=f(0,0)=0$

$f(a,b) \leq f(x,y)$, since f is never negative

so $(0,0)$ is local minmum point.



Lagrange Multipliers

It is sometimes necessary to find the extreme values of a function $f(x, y)$ when its domain is subject to some kind of constraint

Example 1: Find the point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$, that lies closest to the origin.

$|op| = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2 + y^2 + z^2}$

, is the minimum value of the distance which represent the function that subjected to the constraint

$, 2x + y - z - 5 = 0$

$\therefore f(x, y) = x^2 + y^2 + z^2,$

constraint $\quad z = 2x + y - 5$

$h(x, y) = f(x, y, 2x + y - 5) = x^2 + y^2 + (2x + y - 5)^2$

$h_x = 2x + 2(2x + y - 5) \cdot 2$

$\quad = 2x + 8x + 4y - 20 = 0$

$10x + 4y = 20 - - - - - - (1)$

$h_y = 2y + 2(2x + y - 5) \cdot 1$

$\quad = 2y + 4x + 2y - 10 = 0$

$4y + 4x = 10 - - - - - - (2)$

eq (1)& (2) by substrcition , $x = \frac{5}{3}, y = \frac{5}{6}$, and $z = -\frac{5}{6}$

Therefore the closest point , $p\left(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6}\right)$

As we can see the solution by substitution donot always go smoothly.

This one of the reasons for learning the new method

(LAGRANGE MULTIPLER)

Suppose we have $f(x, y, z)$ and $g(x, y, z)$

To find the local maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z)$

$1 -$ Let $g(x, y, z) = 0$

$2 -$ Construct the auxiliary function

$\quad H(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z)$

$3 -$ Then find the values of x, y, z, and λ for which

$\quad H_x = 0, \; H_y = 0, \; H_z = 0, \; H_\lambda = 0$

Where : $\quad H_x = f_x - \lambda g_x = 0, \quad or \quad f_x = \lambda g_x$

$\quad H_y = f_y - \lambda g_y = 0, \quad or \quad f_y = \lambda g_y$

$\quad H_z = f_z - \lambda g_z = 0, \quad or \quad f_z = \lambda g_z$

$\quad H_\lambda = -g(x, y, z) = 0, \quad or \quad g(x, y, z) = 0$

Then find $x, y, z,$ and λ

Partial Derivatives with Constrained Variables

Decide which Variables Are Dependent and Which Are Independent

If the variables in a function $w = f(x, y, z)$ *are constrained by a relation like the one inposed on* x, y *and* z *by the equation* $z = x^2 + y^2$ *, the geometric meaning and the numerical values of the partial derivatives of* f *will depend on which variables are chosen to be dependent and which are chosen to be independent. To see how this choice can affect the outcome, we consider the calculation of* $\partial w/\partial x$ *when* $w = x^2 + y^2 + z^2$ *and* $z = x^2 + y^2$.

Example 1: *Finding a Partial Derivative with Constrained Indep. Var.*
Find $\partial w/\partial x$ *if* $w = x^2 + y^2 + z^2$ *and* $z = x^2 + y^2$
Solution: We are given two equations in the four unknowns $x, y, z,$ *and* w. *Like many such systems, this one can be solved for two of the unknowns (the dependent vars.) in terms of the others (indep. vars.). In being asked for* $\partial w/\partial x$ *, we are told that* w *is to be a dep. var. and* x *an indep. var. The possible choices for the other variables com down to*

References

1. G. B. Thomas, M. D. Weir and J. R. Hass, Thommas' Calculus, 12th Edition, Copyright© 2010, Pearson Education.

2. B.S.Grewal, Higher Engineering Mathematics, Khanna Publishers, 40th Edition, 2007.

Other support books :-

Erwin Kreyszig, Advanced Engineering Mathematics, 8th edition, 2007.