Chapter Five: Higher-Ordered Systems: Second-Ordered and Transportation Lag

Chapter Seven in the textbook

This section introduces a basic system called a second-order system or a quadratic lag. Second-order systems are described by a second-order differential equation that relates the output variable y to the input variable x (the forcing function) with time as the independent variable.

$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = kx(t)$$

A second-order system can arise from two first-order systems in series, as we saw in Chapter 4. Some systems are inherently second-order, and they do not result from a series combination of two first-order systems. A general second-order system under a dynamic condition is given by the differential equation as follows:

$$\frac{1}{\omega_n^2}\frac{d^2Y}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dY}{dt} + Y = kX(t) \qquad \tau = \frac{1}{\omega_n}$$

$$\tau^2\frac{d^2Y}{dt^2} + 2\zeta\tau\frac{dY}{dt} + Y = kX(t)$$

Where k is the steady state gain, Y is the response value, X is the input disturbing variable, ζ is the damping factor (damping coefficient) and ω_n is the natural frequency of oscillation. Taking Laplace for the second-order differential equation,

$$\tau^2 s^2 Y(s) + 2\zeta\tau s Y(s) + Y(s) = kX(s) \Rightarrow (\tau^2 s^2 + 2\zeta\tau s + 1)Y(s) = kX(s)$$

$$G(s) = \frac{Y(s)}{X(s)} = \frac{k}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

If X is a sudden force, such as step change inputs, Y will oscillate depending on the value of the damping coefficient (ζ). If $\zeta < 1$ (Under damped), $\zeta > 1$ (Over damped), $\zeta = 1$ (Critical damped)

Responses of Second Order System

- Step Change

$$\frac{Y(s)}{X(s)} = \frac{k}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

$$X(s) = \frac{A}{s} \quad \Rightarrow \quad Y(s) = \frac{k}{\tau^2 s^2 + 2\zeta\tau s + 1} \times \frac{A}{s} = \frac{k/\tau^2}{s^2 + \frac{2\zeta}{\tau}s + \frac{1}{\tau^2}} \times \frac{A}{s} \qquad (1)$$

The quadratic term in this equation may be factored into two linear terms that contain the roots

$$s_{1,2} = \frac{-\frac{2\zeta}{\tau} \pm \sqrt{\left(\frac{2\zeta}{\tau}\right)^2 - \frac{4}{\tau^2}}}{2} = \frac{-\xi}{\tau} \pm \frac{\sqrt{\frac{4\zeta^2}{\tau^2} - \frac{4}{\tau^2}}}{2} = \frac{-\zeta}{\tau} \pm \frac{2\sqrt{\frac{\zeta^2 - 1}{\tau^2}}}{2}$$

$$= \frac{-\zeta}{\tau} \pm \frac{\sqrt{\zeta^2 - 1}}{\tau} \qquad \text{Two real roots}$$

$$s_1 = \frac{-\zeta}{\tau} - \frac{\sqrt{\zeta^2 - 1}}{\tau} \quad and \quad s_2 = \frac{-\zeta}{\tau} + \frac{\sqrt{\zeta^2 - 1}}{\tau} \qquad (2)$$

Eq. (1) can be re-written as

$$Y(s) = \frac{kA/\tau^2}{s(s - s_1)(s - s_2)}$$

$\zeta > 1$	Overdamped	Two distinct real roots
$\zeta = 1$	Critically Damped	Two equal real roots
$0 < \zeta < 1$	Underdamped	Two complex roots

$$Y(s) = \frac{k}{\tau^2 s^2 + 2\zeta\tau s + 1} \times \frac{A}{s} = \frac{\alpha_0}{s} + \frac{\alpha_1 s + \alpha_2}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

$$\alpha_o(\tau^2 s^2 + 2\zeta\tau s + 1) + \alpha_1 s^2 + \alpha_2 s = kA$$

$$s^0 \qquad \alpha_0 = kA$$

$$s^1 \qquad 2\alpha_0\zeta\tau + \alpha_2 = 0 \quad \Rightarrow \quad \alpha_2 = -2kA\zeta\tau$$

$$s^2 \qquad \alpha_0\tau^2 + \alpha_1 = 0 \quad \Rightarrow \quad \alpha_1 = -kA\tau^2$$

$$Y(s)=kA\left[\frac{1}{s}-\frac{\tau^2 s+2\zeta\tau}{\tau^2 s^2+2\zeta\tau s+1}\right]$$

$$Y(s)=kA\left[\frac{1}{s}-\frac{s+2\frac{\zeta}{\tau}}{\left(s+2\frac{\zeta}{\tau}s+\frac{1}{\tau^2}\right)+\frac{\zeta^2}{\tau^2}-\frac{\zeta^2}{\tau^2}}\right]=kA\left[\frac{1}{s}-\frac{s+2\frac{\zeta}{\tau}}{\left(s+2\frac{\zeta}{\tau}s+\frac{\zeta^2}{\tau^2}\right)+\frac{1}{\tau^2}-\frac{\zeta^2}{\tau^2}}\right]$$

$$Y(s)=kA\left[\frac{1}{s}-\frac{s+2\frac{\zeta}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2+\frac{1-\zeta^2}{\tau^2}}\right]$$

- ζ <1 (Under-damped system)

$$Y(s)=kA\left[\frac{1}{s}-\frac{s+2\frac{\zeta}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2+\left(\frac{\sqrt{1-\zeta^2}}{\tau}\right)^2}\right]=kA\left[\frac{1}{s}-\frac{s+\frac{\zeta}{\tau}+\frac{\zeta}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2+\left(\frac{\sqrt{1-\zeta^2}}{\tau}\right)^2}\right]$$

$$=kA\left[\frac{1}{s}-\frac{s+\frac{\zeta}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2+\left(\frac{\sqrt{1-\zeta^2}}{\tau}\right)^2}-\frac{\frac{\zeta}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2+\left(\frac{\sqrt{1-\zeta^2}}{\tau}\right)^2}\right]$$

$$=kA\left[\frac{1}{s}-\frac{s+\frac{\zeta}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2+\left(\frac{\sqrt{1-\zeta^2}}{\tau}\right)^2}-\frac{\frac{\zeta}{\tau}\times\frac{\tau}{\sqrt{1-\zeta^2}}\times\frac{\sqrt{1-\zeta^2}}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2+\left(\frac{\sqrt{1-\zeta^2}}{\tau}\right)^2}\right]$$

$$Y(t)=kA\left[1-e^{(-\zeta/\tau)t}\cos\frac{\sqrt{1-\zeta^2}}{\tau}t-\frac{\zeta}{\sqrt{1-\zeta^2}}e^{(-\zeta/\tau)t}\sin\frac{\sqrt{1-\zeta^2}}{\tau}t\right]$$

$$w=\frac{\sqrt{1-\zeta^2}}{\tau} \qquad Y(t)=kA\left[1-e^{(-\zeta/\tau)t}\left(\cos wt+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin wt\right]\right.$$

$$r=\sqrt{p^2+q^2}=\sqrt{1+\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)^2}=\sqrt{\frac{1}{1-\zeta^2}}$$

$$\phi=\tan^{-1}\frac{p}{q}=\tan^{-1}\frac{1}{\frac{\zeta}{\sqrt{1-\zeta^2}}}=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$Y(t)=kA\left[1-e^{(-\zeta/\tau)t}(r\sin(wt+\phi))\right]$$

- $\zeta>1$ (Over-damped system)

$$Y(s)=kA\left[\frac{1}{s}-\frac{s+2\frac{\zeta}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2+\frac{1-\zeta^2}{\tau^2}}\right]=kA\left[\frac{1}{s}-\frac{s+\frac{\zeta}{\tau}+\frac{\zeta}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2-\frac{\zeta^2-1}{\tau^2}}\right]$$

$$=kA\left[\frac{1}{s}-\frac{s+\frac{\zeta}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2-\left(\frac{\sqrt{\zeta^2-1}}{\tau}\right)^2}-\frac{\frac{\zeta}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2-\left(\frac{\sqrt{\zeta^2-1}}{\tau}\right)^2}\right]$$

$$=kA\left[\frac{1}{s}-\frac{s+\frac{\zeta}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2-\left(\frac{\sqrt{\zeta^2-1}}{\tau}\right)^2}-\frac{\frac{\zeta}{\tau}\times\frac{\tau}{\sqrt{\zeta^2-1}}\times\frac{\sqrt{\zeta^2-1}}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2-\left(\frac{\sqrt{\zeta^2-1}}{\tau}\right)^2}\right]$$

$$Y(t) = kA\left[1 - e^{-(\zeta/\tau)t}\left(\cosh wt + \frac{\zeta}{\sqrt{\zeta^2 - 1}}\sinh wt\right)\right] \qquad \text{where} \quad w = \frac{\sqrt{\zeta^2 - 1}}{\tau}$$

- $\zeta = 1$ (Critical-damped system)

For this case, the response is given by expression:

$$Y(t) = 1 - \left(1 + \frac{t}{\tau}\right)e^{-t/\tau}$$

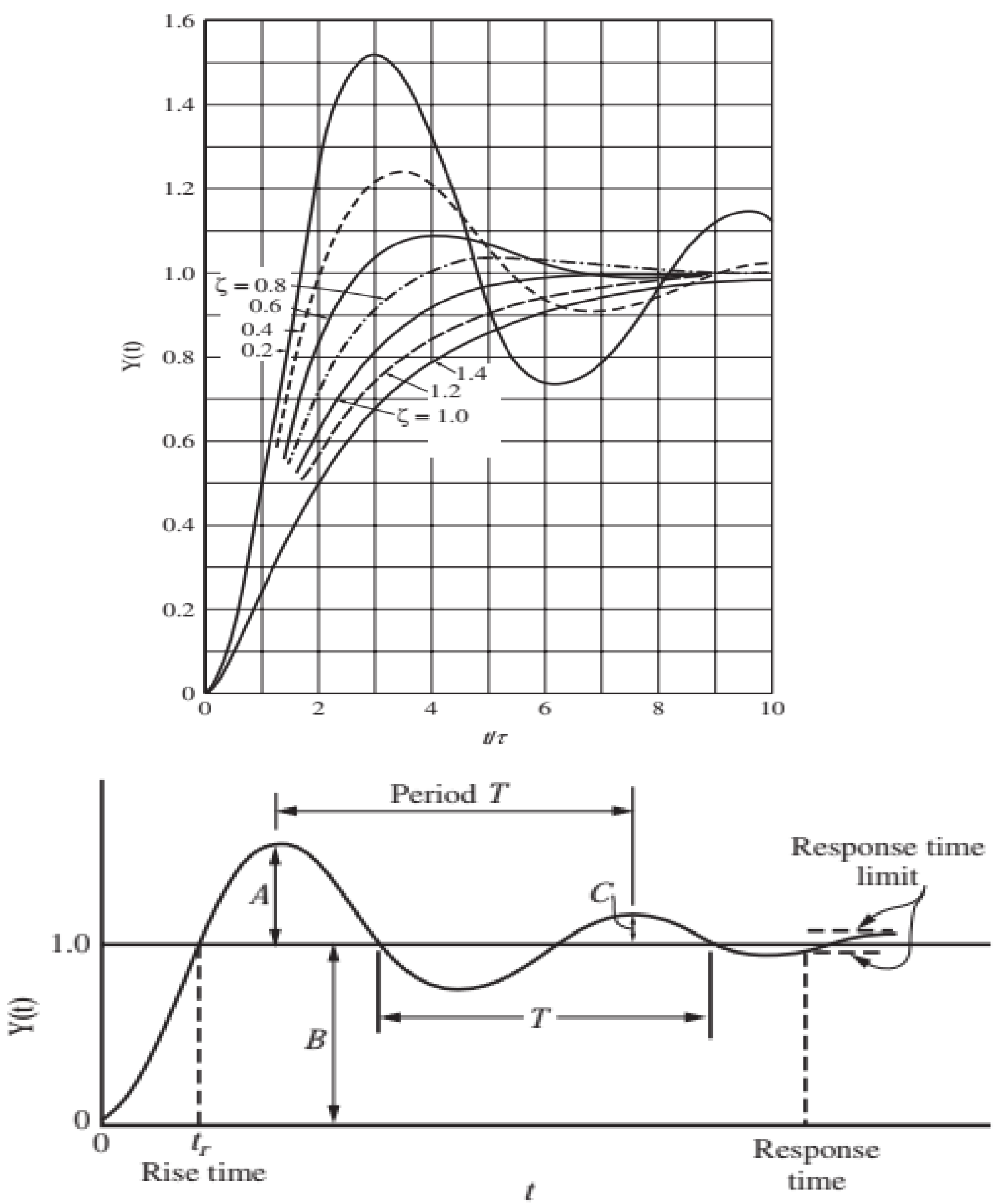


Figure (5.1): Response of the second order system – Under damping.

Terms Used to Describe an Underdamped System Second order system for a step change response

- Overshoot (OS)

It is a measure of how much the response exceeds the ultimate value (new steady-state value) following a step change and is expressed as the ratio $\frac{A}{B}$ in Figure (5.1).

$$OS = \exp \frac{-\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$OS\ \% = OS \times 100$$

$$wt + \phi = \phi + n\pi$$

$$t = \frac{n\pi}{w} \qquad \text{max or min} \quad n = 1, 2, 3$$

If n = 0, 2, 4, 6, ……………. min

If n = 1, 3, 5, 7, ……………. max

1st max when n=1

$$t = \frac{n\pi}{w} = \frac{\pi}{w}$$

$$y(t) = kA\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{\frac{-\zeta\pi}{\tau w}} \sin\left(w\frac{\pi}{w} + \phi\right)\right]$$

$$y_{\max} = kA\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} (-\sin\phi)\right]$$

For the underdampded system

$$\cos\phi = -\zeta \quad , \quad \sin\phi = \sqrt{1-\zeta^2} \quad , \quad \tan\phi = \frac{\sqrt{1-\zeta^2}}{-\zeta}$$

$$y_{\max} = kA\left[1 + \frac{1}{\sqrt{1-\zeta^2}} e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \left(\sqrt{1-\zeta^2}\right)\right]$$

$$y_{\max} = kA\left[1 + e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}\right]$$

$$Overshoot = \frac{A}{B} = \frac{\max - B}{B} = \frac{kA\left[1 + e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}\right] - kA}{kA} = \exp\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}$$

- Decay Ratio (DR)

It is defined as the ratio of the sizes of successive peaks and is given by $\frac{C}{A}$ in Figure (5.1). where C is the height of the second peak.

$$DR = \exp\frac{-2\pi\zeta}{\sqrt{1-\zeta^2}} = (OS)^2$$

$$t = \frac{n\pi}{w} \quad \text{for } n=3 \text{ then } t = \frac{3\pi}{w}$$

First peak at n=1 $y_{\max} = kA\left[1 + e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}\right]$

Second peak at n=3 $y_{\max} = kA\left[1 + e^{\frac{-3\pi\zeta}{\sqrt{1-\zeta^2}}}\right]$

$$Decay\ Ratio = \frac{kA[1 + e^{\frac{-3\pi\zeta}{\sqrt{1-\zeta^2}}}] - kA}{kA[1 + e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}] - kA} = \frac{e^{\frac{-3\pi\zeta}{\sqrt{1-\zeta^2}}}}{e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}} = e^{\frac{-2\pi\zeta}{\sqrt{1-\zeta^2}}}$$

$$Decay\ Ratio = \exp\frac{-2\pi\zeta}{\sqrt{1-\zeta^2}}$$

- Rise Time (t_r)

This is the time required for the response to first reach its ultimate value.

$$t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}}{w}$$

$$y(t) = kA\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{\frac{-\zeta t}{\tau}} \sin(tw + \phi)\right]$$

At t_r y(t) = kA

$$kA = kA[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{\frac{-\zeta t}{\tau}} \sin(t_r w + \phi)]$$

$$0 = \sin(t_r w + \phi)]$$

$$tr = \frac{\sin^{-1}(0) - \phi}{w}$$

$$tr = \frac{n\pi - \phi}{w} = \frac{n\pi - \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}}{w} = \qquad tr = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}}{w} \qquad \text{for n=1}$$

- Response Time

This is the time required for the response to come within ±5 percent of its ultimate value and remain there. The response time is indicated in Figure (5.1).

- Period of Oscillation (T)

The radian frequency (radians/time) is the coefficient of t in the sine term; thus,

$$T = \frac{2\pi\tau}{\sqrt{1-\zeta^2}}$$

$$w = Radian\ frequency = \frac{\sqrt{1-\zeta^2}}{\tau} \qquad \Rightarrow f = \frac{\sqrt{1-\zeta^2}}{2\pi\tau} \qquad \Rightarrow \qquad T = \frac{2\pi\tau}{\sqrt{1-\zeta^2}}$$

$$w = 2\pi f \quad also \quad T = \frac{1}{f}$$

- Natural Period of Oscillation (T_n)

If the damping is eliminated (ζ = 0), the system oscillates continuously without attenuation in amplitude. Under these "natural" or undamped condition, the radian frequency is . This frequency is referred to $\frac{1}{\tau}$ as the natural frequency w_n.

$$w_n = \frac{1}{\tau}$$

The corresponding natural cyclical frequency f_n and period T_n are related by the expression:

$$f_n = \frac{1}{T_n} = \frac{1}{2\pi\tau}$$ Thus, τ has the significance of the undamped period.

The system free of any damping for ζ=0

$$\text{w, radian of frequency} = \frac{\sqrt{1-\zeta^2}}{\tau} \quad \Rightarrow \quad w_n = \frac{1}{\tau} \quad for \quad \zeta = 0$$

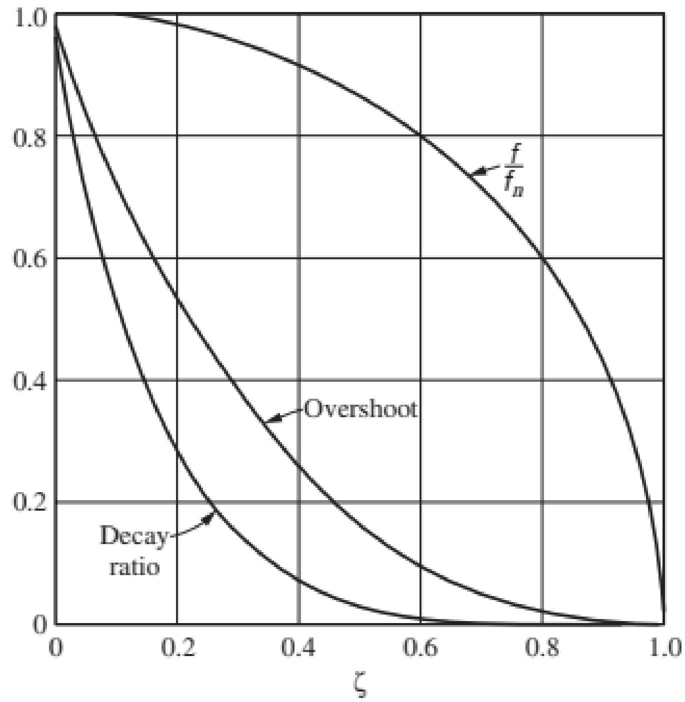
$$w_n = 2\pi f_n \quad \Rightarrow \frac{1}{\tau} = 2\pi f_n \qquad f_n = \frac{1}{2\pi\tau}$$

- Time to First Peak (t_p)

It is the time required for the output to reach its first maximum value.

$$t_p = \frac{\pi}{w} = \frac{\pi\tau}{\sqrt{1-\zeta^2}}$$

$$t = \frac{n\pi}{w}$$ The first peak is reached when n=1 $$t_p = \frac{n\pi}{w} = \frac{\pi}{w} = \frac{\pi\tau}{\sqrt{1-\zeta^2}}$$



- **Impulse Response**

If impulse δ(t) is applied to second order system then transfer response will be as follows:

$$Y(s) = \frac{k}{\tau^2 s^2 + 2\zeta\tau s + 1} X(s)$$

$$X(s) = Area = A$$

$$Y(s) = \frac{k}{\tau^2 s^2 + 2\zeta\tau s + 1} . A$$

$$Y(s) = \frac{kA/\tau^2}{s^2 + \frac{2\zeta}{\tau}s + \frac{1}{\tau^2}} = \frac{kA/\tau^2}{s^2 + \frac{2\zeta}{\tau}s + \frac{1}{\tau^2} + \left(\frac{\zeta}{\tau}\right)^2 - \left(\frac{\zeta}{\tau}\right)^2}$$

$$= \frac{kA/\tau^2}{s^2 + \frac{2\zeta}{\tau}s + \left(\frac{\zeta}{\tau}\right)^2 + \frac{1}{\tau^2} - \left(\frac{\zeta}{\tau}\right)^2} = \frac{kA/\tau^2}{\left(s + \frac{\zeta}{\tau}\right)^2 + \frac{1-\zeta^2}{\tau^2}}$$

- $\zeta > 1$

$$Y(s)=\frac{kA/\tau^2}{(s+\frac{\zeta}{\tau})^2+\frac{1-\zeta^2}{\tau^2}}=\frac{kA/\tau^2}{(s+\frac{\zeta}{\tau})^2-\left(\frac{\sqrt{\zeta^2-1}}{\tau}\right)^2}=\frac{\frac{kA}{\tau^2}\frac{\tau}{\sqrt{\zeta^2-1}}\frac{\sqrt{\zeta^2-1}}{\tau}}{(s+\frac{\zeta}{\tau})^2-\left(\frac{\sqrt{\zeta^2-1}}{\tau}\right)^2}$$

$$Y(t)=\frac{kA}{\tau\sqrt{\zeta^2-1}}e^{\frac{-\zeta t}{\tau}}\sinh wt$$

$$w=\frac{\sqrt{\zeta^2-1}}{\tau}$$

- $\zeta < 1$

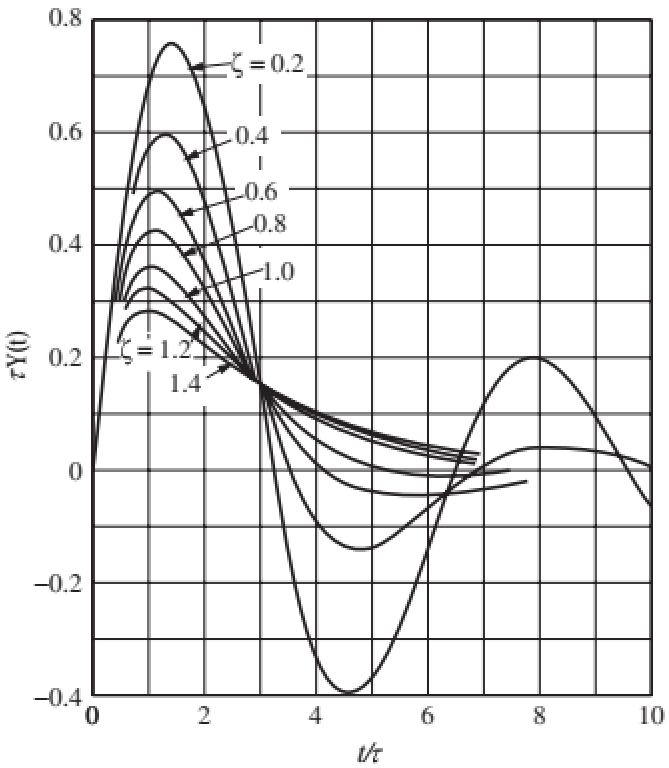
$$Y(s)=\frac{kA/\tau^2}{\left(s+\frac{\zeta}{\tau}\right)^2+\frac{1-\zeta^2}{\tau^2}}=\frac{kA/\tau^2}{\left(s+\frac{\zeta}{\tau}\right)^2+\left(\frac{\sqrt{1-\zeta^2}}{\tau}\right)^2}=\frac{\frac{kA}{\tau^2}\frac{\tau}{\sqrt{1-\zeta^2}}\frac{\sqrt{1-\zeta^2}}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2+\left(\frac{\sqrt{1-\zeta^2}}{\tau}\right)^2}$$

$$Y(t)=\frac{kA}{\tau\sqrt{1-\zeta^2}}e^{\frac{-\zeta t}{\tau}}\sin wt \qquad w=\frac{\sqrt{1-\zeta^2}}{\tau}$$

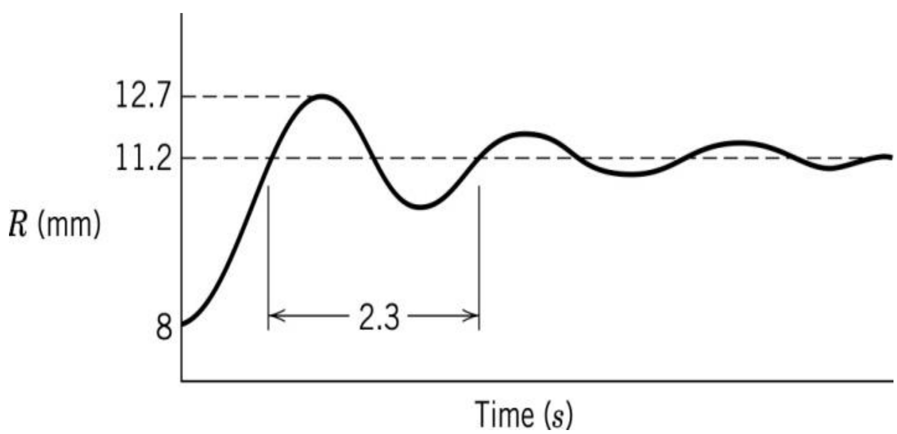
- $\zeta = 1$

$$Y(s)=\frac{kA/\tau^2}{\left(s+\frac{\zeta}{\tau}\right)^2+\frac{1-\zeta^2}{\tau^2}}=\frac{kA/\tau^2}{\left(s+\frac{1}{\tau}\right)^2+\frac{1-1^2}{\tau^2}}=\frac{kA/\tau^2}{\left(s+\frac{1}{\tau}\right)^2}$$

$$Y(t)=\frac{kA}{\tau^2}te^{-t/\tau}$$



Example 1: A step change from 15 to 31 psi in actual pressure results in the measured response from a pressure indicating element shown in the figure beside. Assuming second-order dynamics, calculate all important parameters and write and approximate transfer function in the form



$$\frac{R'(s)}{P'(s)} = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

where R' is the instrument output deviation (mm), P' is the actual pressure deviation (psi).

Solution:

$$Gain = \frac{11.2-8}{31-15} = 0.20 \ mm/psi$$

$$Overshoot = \frac{12.7-11.2}{11.2-8} = 0.47$$

$$Overshoot = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) = 0.47$$

$$\zeta = 0.234$$

$$Period = \frac{2\pi\tau}{\sqrt{1-\zeta^2}} = 2.3\sec \Rightarrow \tau = 2.3 \times \frac{\sqrt{1-0.234^2}}{2\pi} = 0.356\sec$$

$$\frac{R'(s)}{P'(s)} = \frac{0.2}{0.127s^2 + 0.167s + 1}$$

Example 2: A control system having a transfer function is expressed as:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{5}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

The radian frequency for the control system is 1.9 rad/min. The time constant is 0.5 min. The control system is subjected to a step change of the magnitude 2. Calculate: (1) Rise time, (2) Decay ratio, (3) Maximum value of Y(t), and (4) Response time.

Solution:

Given $X(s) = \frac{2}{s}$ Time constant $\tau = 0.5 \min$ Radian frequency $w = 1.9 \ rad/\min$

$$w = \frac{\sqrt{1-\zeta^2}}{\tau} \Rightarrow 1.9 = \frac{\sqrt{1-\zeta^2}}{0.5} \Rightarrow \zeta = 0.312$$

(1) Rise time

$$tr = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}}{w} = \frac{3.1416 - \tan^{-1}\frac{\sqrt{1-0.312^2}}{0.312}}{1.9} = 1.0 \min$$

(2) Deacay ratio $= \frac{C}{A} = \exp\left(\frac{-2\pi\zeta}{\sqrt{1-\zeta^2}}\right) = \exp\left(\frac{-2\pi(0.312)}{\sqrt{1-(0.312)^2}}\right) = 0.127$

(3) Ultimate value of the response $Y_{ultimate}(B)$ at $t \to \infty$

$$\frac{Y(s)}{X(s)} = \frac{5}{0.25s^2 + 0.316s + 1} \qquad X(s) = \frac{2}{s}$$

$$Y(s) = \frac{10}{s(0.25s^2 + 0.316s + 1)}$$

$$\text{Ultimate value} = \lim_{s\to 0} sY(s) \rightarrow \lim_{s\to 0} s\left(\frac{10}{0.25s^2 + 0.316s + 1}\right) = 10$$

$Y_{ultimate}(B) = 10$ Maximum value of response $= B\left(1 + \frac{B}{A}\right)$

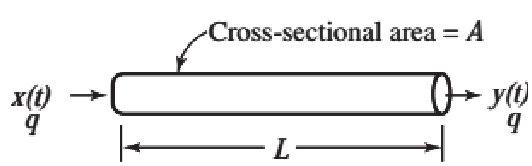
Overshoot $= \frac{B}{A} = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)$ Decay ratio=Overshoot^2

$0.127 = \text{Overshoot}^2 \Rightarrow \text{Overshoot} = 0.356 = \frac{B}{A} \Rightarrow$ Maximum value of response $=10(1+0.356)=13.56$

(4) Response time ts $=3\frac{\tau}{\zeta} = 4.8077$ min for $\pm 5\%$ *of ultimate value*

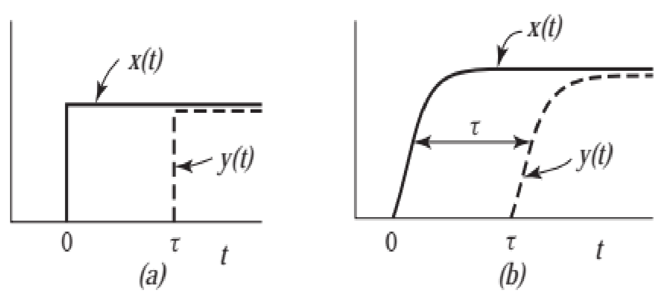
Transportation Lag

A phenomenon that is often present in flow systems is the transportation lag. Synonyms for this term are dead time and distance velocity lag. As an example, consider the system shown in the figure beside, in which a liquid flows through an insulated tube of uniform cross sectional area A and length L at a constant volumetric flow rate q. The density r and the heat capacity C are constant. The tube wall has a negligible heat capacity, and the velocity profile is flat (plug flow). The temperature x of the



entering fluid varies with time, and it is desired to find the response of the outlet temperature y (t) in terms of a transfer function.

If a step change were made in x (t) at t 0, the change would not be detected at the end of the tube until t_ds later, where t_d is the time required for the entering fluid to pass through the tube. This simple step response is shown in the figure beside. If the variation in x(t) were some arbitrary function, as shown in the figure beside, the response y(t) at the end of the pipe would be identical with x(t) but again delayed by t_d units of time. The transportation lag parameter t_d is simply the time needed for a particle of fluid to flow from the entrance of the tube to the exit, and it can be calculated from the expression:



$$t_d = \frac{Volume\ of\ Tube}{Volumatric\ Flow\ rate} = \frac{AL}{V}$$

It can be seen from the figure above that the relationship between y (t) and x(t) is

$$y(t) = x(t-\tau)$$

Introducing the deviation variables $X = x - x_s\ and\ Y = y - y_s$

$$Y(t) = X(t-\tau)$$

Taking the Laplace transform, $Y(s) = X(s)e^{-t_d s} \rightarrow \frac{Y(s)}{X(s)} = e^{-t_d s}$

Therefore, the transfer function of a transportation lag is $e^{-t_d s}$

Approximation of a transportation lag as follows:

Taylor series: $e^{-t_d s} \cong \frac{1}{1+t_d s}$

$$e^{-t_d s} = \frac{e^{-t_d s/2}}{e^{t_d s/2}}$$

first-order Padé approximation: $e^{-t_d s} = \frac{1-t_d s/2}{1+t_d s/2}$

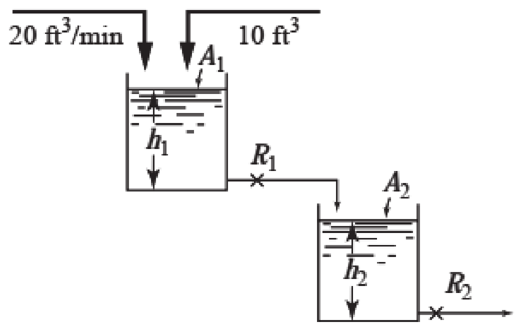
Problems

7.1. A step change of magnitude 4 is introduced into a system having the transfer function

$$\frac{Y(s)}{X(s)} = \frac{10}{s^2 + 1.6s + 4}$$

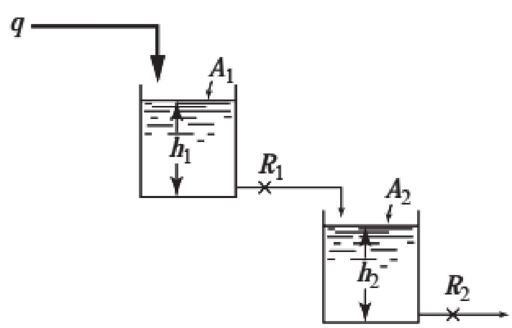
Determine (a) Percent overshoot, (b) Rise time (c), Maximum value of Y (t), (d) Ultimate value of Y (t) and (e) Period of oscillation.

7.2. The two-tank system shown in the figure beside is operating at a steady state. At time, t = 0, 10 ft^3 of water is quickly added to the first tank. Using appropriate figures and equations in the text, determine the maximum deviation in level (feet) in both tanks from the ultimate steady-state values and the time at which each maximum occurs. Data:



$A_1 = A_2 = 10$ ft^2, $R_1 = 0.1$ ft/cfm, $R_2 = 0.35$ ft/cfm

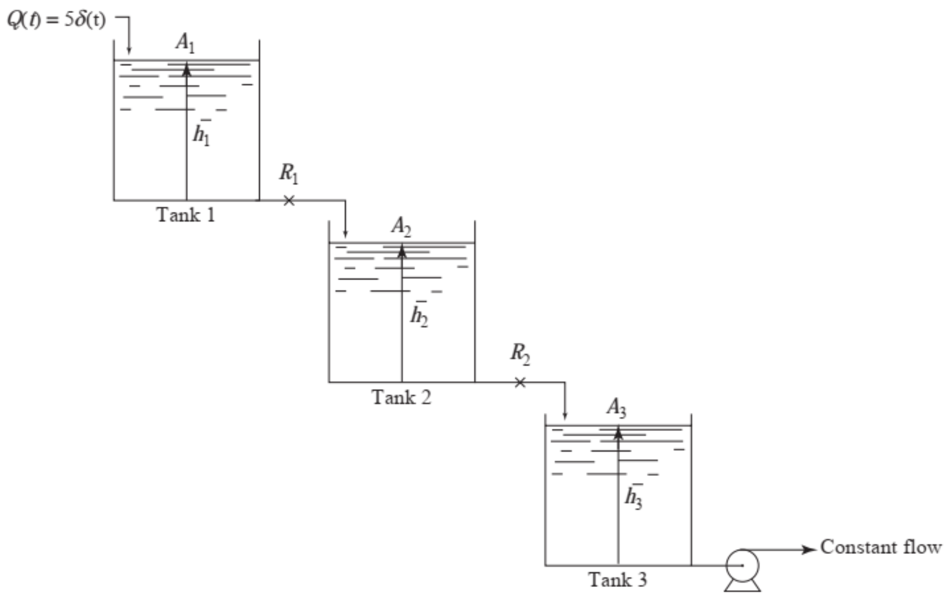
7.3. The two-tank liquid-level system shown in the figure beside is operating at steady state when a step change is made in the flow rate to tank 1. The transient response is critically damped, and it takes 1.0 min for the change in level of the second tank to reach 50 percent of the total change. If the ratio of the cross-sectional areas of the tanks is $A_1/A_2 = 2$, calculate the ratio R_1/R_2. Calculate the time constant for each tank. How long does it take for the change in the level of the first tank to reach 90 percent of the total change?



7.10. Determine Y (0), Y (0.6), and Y (∞) if

$$Y(s) = \frac{1}{s}\frac{25(s+4)}{s^2 + 2s + 25}$$

7.11. In the liquid-level system shown in the figure beside, the deviation in flow rate to the first tank is an impulse function of magnitude 5. The following data apply: $A_1 = 1$ ft^2, $A_2 = A_3 = 2$ ft^2, $R_1 = 1$ ft /cfm, and $R^2 = 1.5$ ft/cfm.



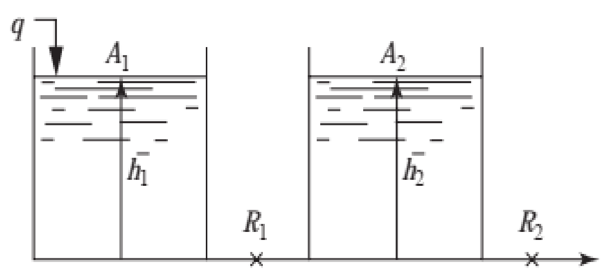
a) Determine expressions for $H_1(s)$, $H_2(s)$, and $H_3(s)$ where H_1, H_2, and H_3 are deviations in tank level for tanks 1, 2, and 3.

b) Sketch the responses of $H_1(t)$, $H_2(t)$, and $H_3(t)$. (You need to show only the shape of the responses; do not plot.)

c) Determine $H_1(3.46)$, $H_2(3.46)$, and $H_3(3.46)$. For H_2 and H_3, use graphs in Chap. 7 of this text after first finding values of t and z for an equivalent second-order system.

7.12. Sketch the response Y(t) if $Y(s) = \frac{e^{-2s}}{[s^2+1.2s+1]}$. Determine Y(t) for t = 0, 1, 5, and ∞.

7.13. The two tanks shown in the figure beside are connected in an interacting fashion. The system is initially at a steady state with q = 10 cfm. The following data apply to the tanks: $A_1 = 1$ ft^2, $A_2 = 1.25$ ft^2, $R_1 = 1$ ft/cfm, and $R_2 = 0.8$ ft/cfm.



a) If the flow changes from 10 to 11 cfm according to a step change, determine $H_2(s)$, i.e., the Laplace transform of $H_2(t)$, where H_2 is the deviation in h_2.

b) Determine $H_2(1)$, $H_2(4)$, and $H_2(\infty)$.

c) Determine the initial levels (actual levels) $h_1(0)$ and $h_2(0)$ in the tanks.

d) Obtain an expression for $H_1(s)$ for the unit-step change described above.

7.14. From figures in this chapter, determine Y(4) for the system response expressed by

$$Y(s) = \frac{2}{s}\frac{2s+4}{s^2+0.8s+1}$$

7.15. A step change of magnitude 3 is introduced into the transfer function

$$Y(s) = \frac{10}{2s^2+0.3s+0.5}$$

Determine the overshoot and the frequency of oscillation.