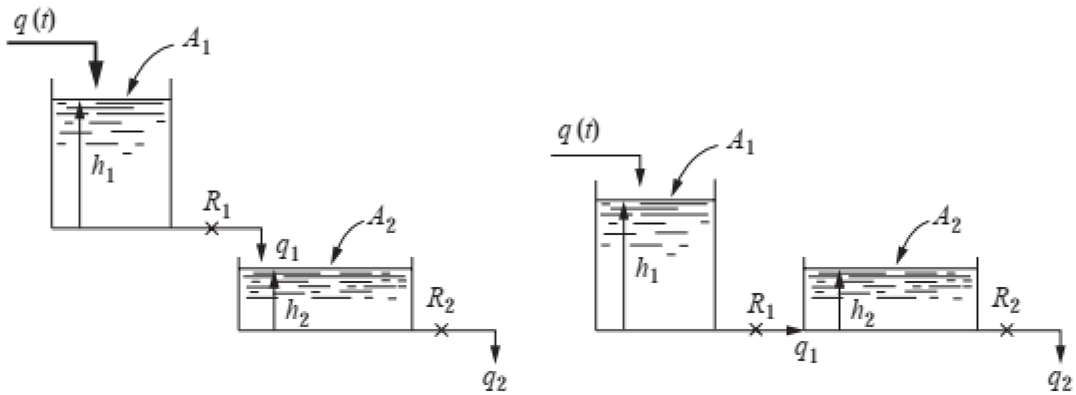
Chapter Four: Response of First Order-Order Systems in Series

Chapter Six in the textbook

A physical system can be represented by several first-order processes connected in series. Consider the liquid-level systems shown in the figure below in which two tanks are arranged so that the outlet flow from the first tank is the inlet flow to the second tank. Two possible piping arrangements are shown in in the figure below. In the figure a, the outlet flow from Tank 1 discharges directly into the atmosphere before spilling into tank 2, and the flow through R_1 depends only on h_1. The variation in h_2 in Tank 2 does not affect the transient response occurring in Tank 1. This type of system is referred to as a noninteracting system. In contrast to this, the system shown in the figure b is said to be interacting because the flow through R_1 now depends on the difference between h_1 and h_2.



(a) (b)

Non-interacting System

A balance on Tank 1 gives:

$$q - q_1 = A_1 \frac{dh_1}{dt} \quad (1)$$

$$q_s - q_{1s} = A_1 \frac{dh_{1s}}{dt} \quad (2)$$

$$(q - q_s) - (q_1 - q_{1s}) = A_1 \frac{d(h_1 - h_{1s})}{dt} \quad (3)$$

$$(\mathrm{q}-\mathrm{q_s})=(\mathrm{q_1}-\mathrm{q_{1s}})+A_1\frac{\mathrm{d}(h_1-h_{1s})}{\mathrm{dt}} \quad (4)$$

In a deviation form, $\mathrm{Q}=\mathrm{q}-\mathrm{q_s}$ $\mathrm{Q_1}=\mathrm{q_1}-\mathrm{q_{1s}}$ Taking the Laplace transform,

$$\frac{\mathrm{Q_1(s)}}{\mathrm{Q(s)}}=\frac{1}{\tau_1\mathrm{s}+1} \quad (5)$$

$\tau_1=\mathrm{A_1R_1}$ for a level Tank 1.

A balance on Tank 2 gives:

$$\mathrm{q_1}-\mathrm{q_2}=A_2\frac{\mathrm{d}h_2}{\mathrm{dt}} \quad (6)$$

$$\mathrm{q_{1s}}-\mathrm{q_{2s}}=A_2\frac{\mathrm{d}h_{1s}}{\mathrm{dt}} \quad (7)$$

$$(\mathrm{q_1}-\mathrm{q_{1s}})-(\mathrm{q_2}-\mathrm{q_{2s}})=A_2\frac{\mathrm{d}(h_2-h_{2s})}{\mathrm{dt}} \quad (8)$$

$$(\mathrm{q_1}-\mathrm{q_{1s}})=(\mathrm{q_2}-\mathrm{q_{2s}})+A_2\frac{\mathrm{d}(h_2-h_{2s})}{\mathrm{dt}} \quad (9)$$

In a deviation form, $\mathrm{Q_1}=\mathrm{q_1}-\mathrm{q_{1s}}$ $\mathrm{Q_2}=\mathrm{q_2}-\mathrm{q_{2s}}$ Taking the Laplace transform,

$$\frac{Q_2(\mathrm{s})}{Q_1(\mathrm{s})}=\frac{1}{\tau_2\mathrm{s}+1} \quad (10)$$

$\tau_2=\mathrm{A_2R_2}$ for a level Tank 2.

$$Q_2(\mathrm{s})=\frac{H_2(\mathrm{s})}{R_2} \quad (11)$$

$$\frac{H_2(\mathrm{s})}{Q_1(\mathrm{s})}=\frac{R_2}{\tau_2\mathrm{s}+1} \quad (12)$$

Sub. Eq. 5 into Eq. 12:

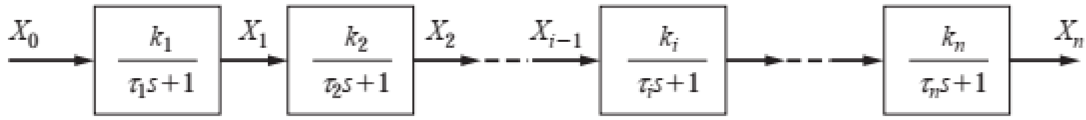
$$\frac{H_2(\mathrm{s})}{\mathrm{Q(s)}}=\frac{1}{\tau_1\mathrm{s}+1}\frac{R_2}{\tau_2\mathrm{s}+1} \quad \text{or} \quad \frac{H_2(\mathrm{s})}{\mathrm{Q(s)}}=\frac{R_2}{\tau_1\tau_2s^2+(\tau_1+\tau_2)\mathrm{s}+1} \quad (13)$$

$\mathrm{Q(s)}=\frac{1}{s}$ Taking Laplace inverse,

$$H_2(\mathrm{t})=R_2\left[\frac{\tau_1\tau_2}{\tau_1-\tau_2}\left(\frac{1}{\tau_2}e^{-t/\tau_1}-\frac{1}{\tau_1}e^{-t/\tau_2}\right)\right] \quad (14)$$

Generalization for Several Noninteracting Systems in Series

We have observed that the overall transfer function for two noninteracting first-order systems connected in series is simply the product of the individual transfer functions. We may now generalize this concept by considering *n* noninteracting first-order systems as represented by the block diagram of the following figure.

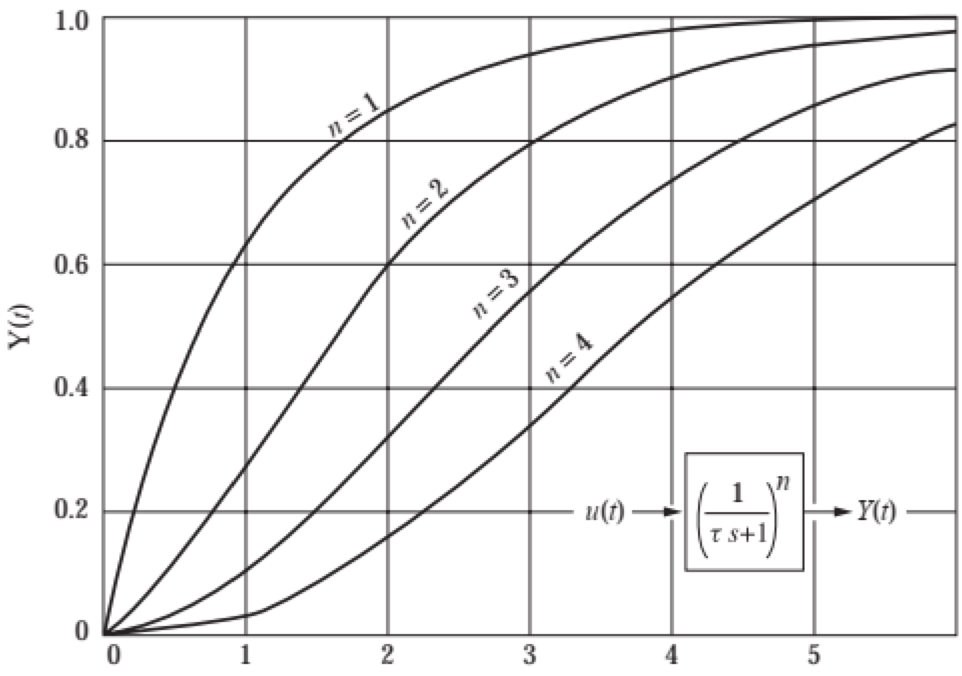


The block diagram is equivalent to the relationships

$$\frac{X_1(s)}{X_o(s)} = \frac{K_1}{\tau_1 s+1} \qquad \frac{X_2(s)}{X_1(s)} = \frac{K_2}{\tau_2 s+1} \qquad \frac{X_3(s)}{X_2(s)} = \frac{K_3}{\tau_3 s+1} \qquad \frac{X_n(s)}{X_{n-1}(s)} = \frac{K_n}{\tau_n s+1}$$

To obtain the overall transfer function, we simply multiply the individual transfer functions; thus

$$\frac{X_n(s)}{X_o(s)} = \prod_{i=1}^{n} \frac{K_i}{\tau_i s + 1}$$



Interacting System

To illustrate an interacting system, we will derive the transfer function for the system shown in the figure b.

Tank 1: $q - q_1 = A_1 \frac{dh_1}{dt}$ (1)

Tank 2: $q_1 - q_2 = A_2 \frac{dh_2}{dt}$ (2)

Tank 1: $q_1 = \frac{h_1 - h_2}{R_1}$ (3)

Tank 2: $q_2 = \frac{h_2}{R_2}$ (4)

At a steady state,

Tank 1: $q_s - q_{1_s} = 0$ (5)

Tank 2: $q_{1_s} - q_{2_s} = 0$ (6)

In a deviation form,

Tank 1: $Q - Q_1 = A_1 \frac{dH_1}{dt}$ (7)

Tank 2: $Q_1 - Q_2 = A_2 \frac{dH_2}{dt}$ (8)

Valve 1: $Q_1 = \frac{H_1 - H_2}{R_1}$ (9)

Valve 2: $Q_2 = \frac{H_2}{R_2}$ (10)

Taking Laplace and Rearrange,

Tank 1: $Q(s) - Q_1(s) = A_1 s H_1(s)$ (11)

Tank 2: $Q_1(s) - Q_2(s) = A_2 s H_2(s)$ (12)

Valve 1: $R_1 Q_1(s) = H_1(s) - H_2(s)$ (13)

Valve 2: $R_2 Q_2(s) = H_2(s)$ (14)

$$\frac{H_2(\mathrm{s})}{\mathrm{Q(s)}} = \frac{R_2}{\tau_1\tau_2 s^2+(\tau_1+\tau_2+A_1R_2)\mathrm{s}+1} \quad (15)$$

The term interacting is often referred to as loading. The second tank of the figure b is said to load the first tank. To understand the effect of interaction on the transient response of a system, consider a two-tank system for which the time constants are equal ($\tau_1 = \tau_2 = \tau$).

If the tanks are noninteracting, the transfer function relating inlet flow to outlet flow is:

$$\frac{Q_2(\mathrm{s})}{\mathrm{Q(s)}} = \left(\frac{1}{\tau s + 1}\right)^2$$

The unit-step response for this transfer function can be obtained by the usual procedure to give

$$Q_2(\mathrm{t}) = 1 - e^{-t/\tau} - \frac{t}{\tau} e^{-t/\tau}$$

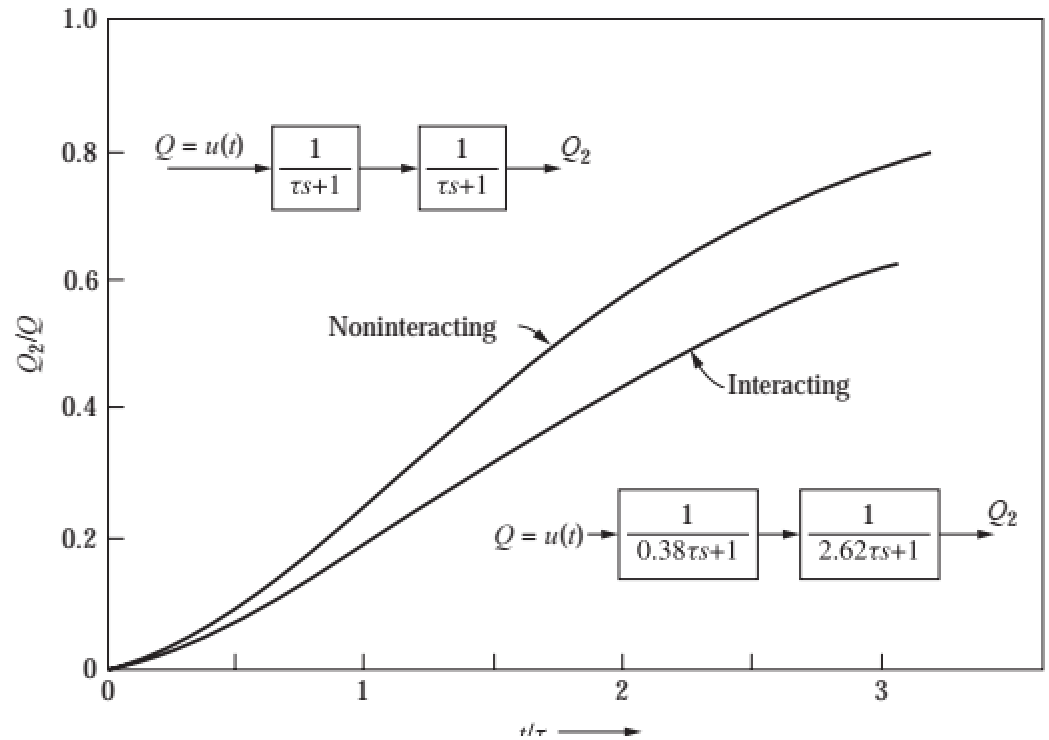
If the tanks are interacting, the overall transfer function, according to Eq. (6.24), is (assuming further that $(A_1 = A_2)$

$$\frac{Q_2(\mathrm{s})}{\mathrm{Q(s)}} = \frac{R_2}{\tau^2 s^2 + 3\tau\mathrm{s} + 1}$$

By application of the quadratic formula, the denominator of this transfer function can be written as:

$$\frac{Q_2(\mathrm{s})}{\mathrm{Q(s)}} = \frac{1}{(0.38\tau\mathrm{s} + 1)(2.62\tau\mathrm{s} + 1)}$$

$$Q_2(\mathrm{t}) = 1 + 0.17e^{-t/0.38\tau} - 1.17e^{-t/2.62\tau}$$



In general, the effect of interaction on a system containing two first-order lags is to change the ratio of effective time constants in the interacting system. In terms of the transient response, this means that the interacting system is more sluggish than the noninteracting system.

Example 6.1. Two noninteracting tanks are connected in series as shown in Fig. 6–1*a*. The time constants are $\tau_2 = 1$ and $\tau_1 = 0.5$; $R_2 = 1$. Sketch the response of the level in tank 2 if a unit-step change is made in the inlet flow rate to tank 1.

The transfer function for this system is found directly from Eq. (6.7); thus

$$\frac{H_2(s)}{Q(s)} = \frac{R_2}{(\tau_1 s + 1)(\tau_2 s + 1)} \tag{6.8}$$

For a unit-step change in Q, we obtain

$$H_2(s) = \frac{1}{s}\frac{R_2}{(\tau_1 s + 1)(\tau_2 s + 1)} \tag{6.9}$$

Inversion by means of partial fraction expansion gives

$$H_2(t) = R_2\left[1 - \frac{\tau_1\tau_2}{\tau_1 - \tau_2}\left(\frac{1}{\tau_2}e^{-t/\tau_1} - \frac{1}{\tau_1}e^{-t/\tau_2}\right)\right] \tag{6.10}$$

Substituting in the values of τ_1, τ_2, and R_2 gives

$$H_2(t) = 1 - \left(2e^{-t} - e^{-2t}\right) \tag{6.11}$$

A plot of this response is shown in Fig. 6–2. Notice that the response is S-shaped and the slope dH_2/dt at the origin is zero. If the change in flow rate were introduced into the second tank, the response would be first-order and is shown for comparison in Fig. 6–2 by the dotted curve.

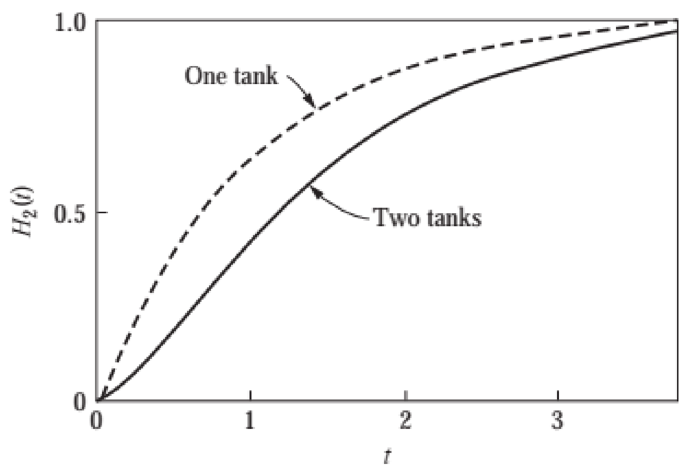
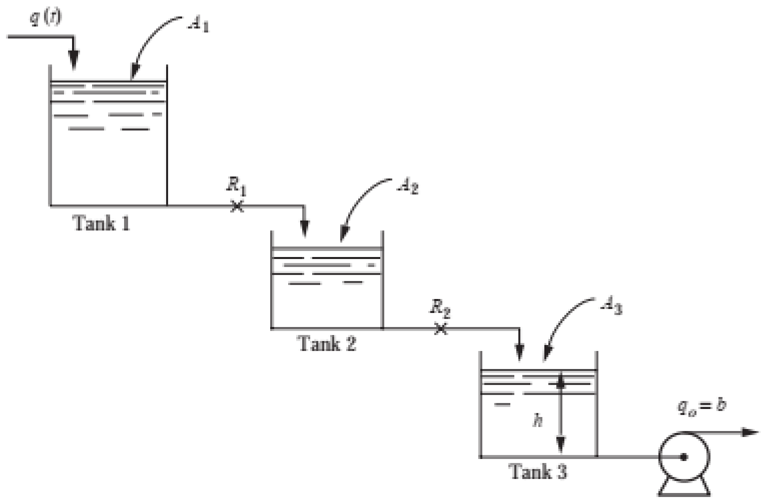


FIGURE 6–2
Transient response of liquid-level system (Example 6.1).

Problems

6.1. Determine the transfer function H(s)/Q(s) for the liquid-level system shown in the following figure. Resistances R_1 and R_2 are linear. The flow rate from tank 3 is maintained constant at b by means of a pump; i.e., the flow rate from tank 3 is independent of head h. The tanks are non-interacting.



6.2. The mercury thermometer in Chap. 4 was considered to have all its resistance in the convective film surrounding the bulb and all its capacitance in the mercury. A more detailed analysis would consider both the convective resistance surrounding the bulb and that between the bulb and mercury. In addition, the capacitance of the glass bulb would be included. Let

Ai = inside area of bulb, for heat transfer to mercury, A0 = outside area of bulb, for heat transfer from surrounding fluid

m = mass of mercury in bulb, rnb = mass of glass bulb

C = heat capacity of mercury, Cb = heat capacity of glass bulb

hi = convective coefficient between bulb and mercury, h, = convective coefficient between bulb and surrounding fluid

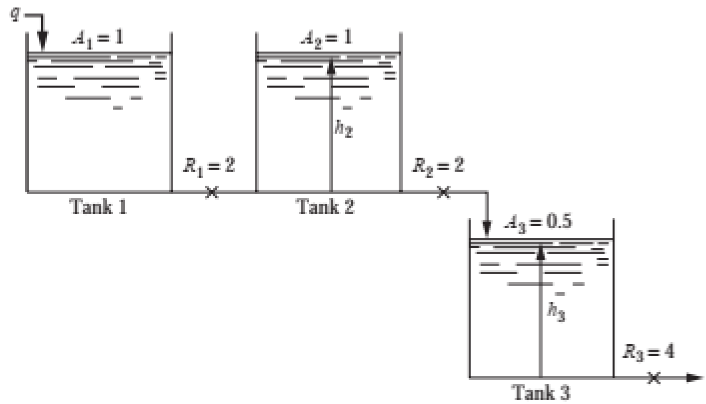
T = temperature of mercury, Tb = temperature of glass bulb

Tf = temperature of surrounding fluid

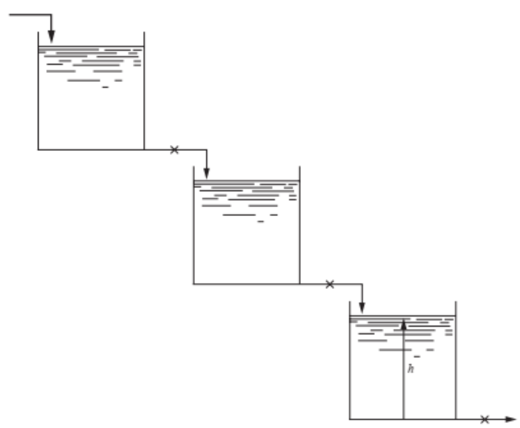
Determine the transfer function between T_f and T. What is the effect of the bulb resistance and capacitance on the thermometer response? Note that the inclusion of the bulb results in a pair of interacting systems, which give an overall transfer function.

6.3. There are N storage tanks of volume V arranged so that when water is fed into the first tank, an equal volume of liquid overflows from the first tank into the second tank, and so on. Each tank initially contains component A at some concentration C_o and is equipped with a perfect stirrer. At time zero, a stream of zero concentration is fed into the first tank at a volumetric rate q. Find the resulting concentration in each tank as a function of time.

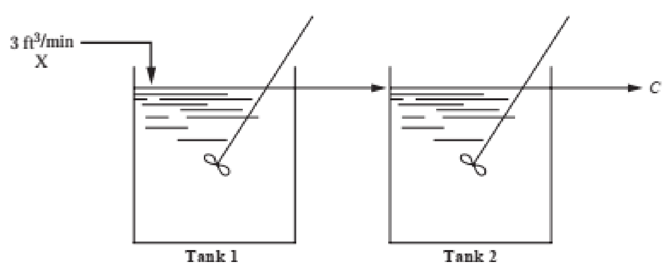
6.4. (a) Find the transfer functions $H_2/Q(s)$ and H_3/Q for the three-tank system shown in the following figure where: H_2, H_3 and Q are deviation variables. Tank 1 and Tank 2 are interacting. (b) For a unit-step change in q (i.e., Q = l/s), determine $H_3(0)$, $H_3(\infty)$, and sketch $H_3(t)$ versus t.



6.5. Three identical tanks are operated in series in a non-interacting fashion as shown in the following figure. For each tank, R = 1, τ = 1. If the deviation in flow rate to the first tank is an impulse function of magnitude 2, determine (a) An expression for H(s) where H is the deviation in level in the third tank. (b) Sketch the response H (t). (c) Obtain an expression for H (t).



6.6. In the two-tank mixing process shown in the following figure, x varies from 0 lb salt/ft^3 to 1 lb salt/ft^3 according to a step function. At what time does the salt concentration in tank 2 reach 0.6 lb salt/ft^3? The holdup volume of each tank is 6 ft^3.



6.7. Starting from first principles, derive the transfer functions $H_1(s)/Q(s)$ and $H_2(s)/Q(s)$ for the liquid level system shown in the following figure. The resistances are linear and R_1 = R_2 =1. Note that two streams are flowing from tank 1, one of which flows into tank 2. You are expected to give numerical values of the parameters in the transfer functions and to show clearly how you derived the transfer functions.