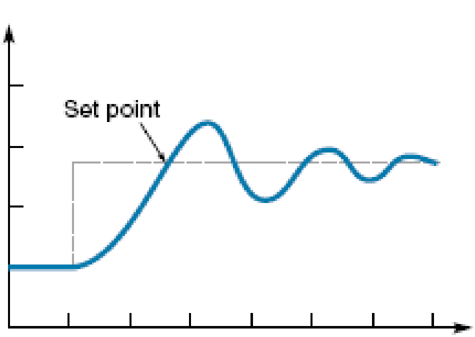
# Chapter Four: Stability

Chapter Thirteen in the textbook

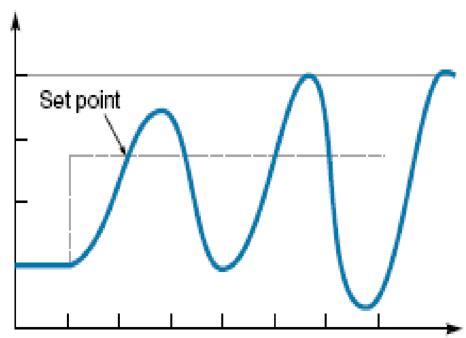
A stable system will be defined as one for which the output response is bounded for all bounded inputs. A system exhibiting an unbounded response to a bounded input is unstable.

$$\text{Stable system} \rightarrow a\ bounded\ input\ produces\ a\ bounded\ output\ (BIBO)$$

A bounded input function is a function of time that always falls within certain bounds over time. For example, the step function and sinusoidal function are bounded inputs. The function $f(t) = t$ is unbounded.

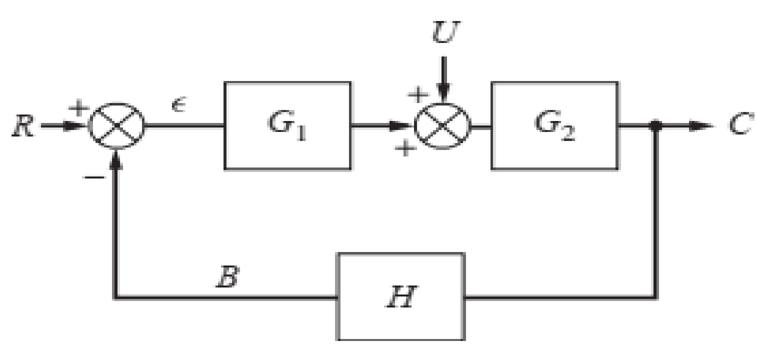



Stable system




Unstable system

## Characteristic Equation




From the block diagram of the control system, the overall transfer function is:

$$C = \frac{G_1 G_2}{1 + G_1 G_2 H} R + \frac{G_2}{1 + G_1 G_2 H} U$$

Characteristic Equation is $1 + G_1 G_2 H = 0 \quad or \quad 1 + G_{loop} = 0$

where $1 + G_{loop}$ is the denominator of the closed-loop transfer function for the process.

It is important to note that the characteristic equation of a control system, which determines its stability, is the same for the set point or load changes. Although the rules derived above were based on a step input, they apply to any input.

**Example 1:** A control system has transfer functions. Find the characteristic equation and its roots, and determine whether the system is stable.

$G_1 = 10\frac{0.5s+1}{s}$ (PI control) $G_2 = \frac{1}{2s+1}$ (Process) $H = 1$ (measuring element with lag)

Characteristic Equation is

$$1 + G_1 G_2 H = 0 \rightarrow 1 + 10\frac{0.5s+1}{s} \times \frac{1}{2s+1} \times 1 = 0 \rightarrow 1 + \frac{5s+10}{s(2s+1)} = 0 \rightarrow \frac{s(2s+1)}{s(2s+1)} + \frac{5s+10}{s(2s+1)} = 0$$

$$s(2s+1) + 5s + 10 = 0 \rightarrow 2s^2 + s + 5s + 10 = 0 \rightarrow 2s^2 + 6s + 10 = 0$$

$$s^2 + 3s + 5 = 0$$

By solving the equation above, the roots are as follows:

$$s = -\frac{3}{2} \mp \frac{\sqrt{9-20}}{2} \rightarrow s1 = -\frac{3}{2} - j\frac{\sqrt{11}}{2} \; and \; s2 = -\frac{3}{2} + j\frac{\sqrt{11}}{2}$$

since the real part of s1 and s2 is negative $(-\frac{3}{2})$, the system is stable.

**Routh Test**

The Routh test is a purely algebraic method for determining how many roots of the characteristic equation have positive real parts; from this, it can also be determined whether the system is stable, for if there are no roots with positive real parts, the system is stable. The test is limited to systems that have polynomial characteristic equations. This means that it cannot be used to test the stability of a control system containing a transportation lag.

The procedure for examining the roots is to write the characteristic equation in the form:

$$a_0s^n + a_1s^{n-1} + a_2s^{n-2} + \cdots + a_n = 0$$

where $a_0$ is positive. (If $a_0$ is originally negative, both sides are multiplied by -1.) In this form, all the coefficients must be positive if all the roots are to lie in the left half-plane. If all the coefficients $a_0, a_1, a_2 \; and \; a_n$ are positive, the system may be stable or unstable. If any coefficient is negative, the system is definitely unstable, and the Routh test is not needed to answer the question of stability.

| Row | | | | |
|---|---|---|---|---|
| 1 | $a_0$ | $a_2$ | $a_4$ | $a_6$ |
| 2 | $a_1$ | $a_3$ | $a_5$ | $a_7$ |
| 3 | $b_1$ | $b_2$ | $b_3$ | |
| 4 | $c_1$ | $c_2$ | $c_3$ | |
| 5 | $d_1$ | $d_2$ | | |
| 6 | $e_1$ | $e_2$ | | |
| 7 | $f_1$ | | | |
| $n+1$ | $g_1$ | | | |

$$b_1 = \frac{a_1a_2 - a_0a_3}{a_1} \qquad b_2 = \frac{a_1a_4 - a_0a_5}{a_1} \qquad b_3 = \frac{a_1a_6 - a_0a_7}{a_1}$$

$$c_1 = \frac{b_1a_3 - a_1b_2}{b_1} \qquad c_2 = \frac{b_1a_5 - a_1b_3}{b_1}$$

$$d_1 = \frac{c_1b_2 - b_1c_2}{c_1} \qquad d_2 = \frac{c_1b_3 - b_1c_3}{c_1}$$

Examine the elements of the first column of the array $a_0, a_1, b_1, c_1$ *and* $d_1$.

1. If any of these elements are negative, we have at least one root on the right of the imaginary axis and the system is unstable.
2. The number of sign changes in the elements of the first column is equal to the number of roots to the right of the imaginary axis.

So, the system is stable if all the elements in the first column of the array are positive.

**Example 2:** Given the characteristic equation:

$$s^4 + 3s^3 + 5s^2 + 4s + 2 = 0$$

determine the stability by the Routh criterion.

Since all the coefficients are positive, the system may be stable. To test this, form the following Routh array:

| Row | | | |
|---|---|---|---|
| 1 | 1 | 5 | 2 |
| 2 | 3 | 4 | |
| 3 | 11/3 | 6/3 | |
| 4 | 26/11 | 0 | |
| 5 | 2 | | |

$$b_1 = \frac{a_1 a_{2-} a_0 a_3}{a_1} = \frac{3 \times 5 - 1 \times 4}{3} = \frac{11}{3}$$

$$b_2 = \frac{a_1 a_{4-} a_0 a_5}{a_1} = \frac{3 \times 2 - 1 \times 0}{3} = \frac{6}{3}$$

$$c_1 = \frac{b_1 a_{3-} a_1 b_2}{b_1} = \frac{\frac{11}{3} \times 4 - 3 \times \frac{6}{3}}{\frac{11}{3}} = \frac{26}{11}$$

Since there is no change in the sign in the first column, no roots are having positive real parts, and the system is stable.

**Problems**

**13.1.** Write the characteristic equation and construct the Routh array for the control system shown in Fig. P13–1. Is the system stable for (*a*) $K_c = 9.5$, (*b*) $K_c = 11$, and (*c*) $K_c = 12$?

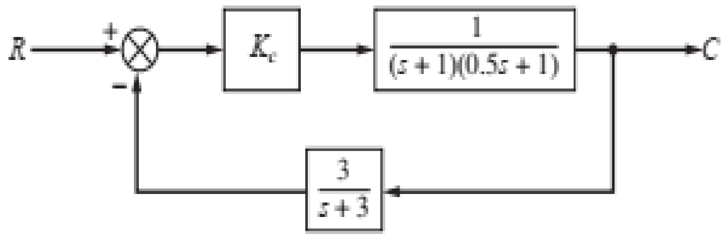



**FIGURE P13–1**

**13.2.** By means of the Routh test, determine the stability of the system shown in Fig. P13–2 when $K_c = 2$.

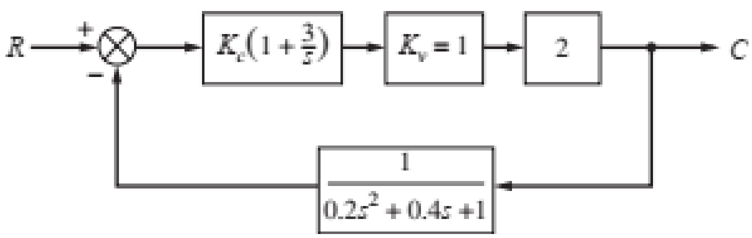



**FIGURE P13–2**

**13.3.** In the control system of Prob. 12.6, determine the value of gain (psi/°F) that just causes the system to be unstable if (*a*) $\tau_D = 0.25$ min and (*b*) $\tau_D = 0.5$ min.

**13.4.** Prove that if one or more of the coefficients $a_0, a_1, \ldots, a_n$ of the characteristic equation [Eq. (13.9)] is negative or zero, then there is necessarily an unstable root. *Hint:* First show that $a_1/a_0$ is minus the sum of all the roots, $a_2/a_0$ is plus the sum of all possible products of two roots, $a_j/a_0$ is $(-1)^j$ times the sum of all possible products of j roots, etc.

**13.5.** Prove that the converse statement of Prob. 13.4—that an unstable root implies that one or more of the coefficients will be negative or zero—is untrue for all $n > 2$. *Hint:* To prove that a statement is untrue, it is only necessary to demonstrate a single counterexample.

**13.6.** Deduce an extension of the Routh criterion that will detect the presence of roots with real parts greater than $-\sigma$ for any specified $\sigma > 0$.

**13.7.** Show that any complex number $s$ satisfying $|s| < 1$ yields a value of

$$z = \frac{1+s}{1-s}$$

that satisfies

$$\text{Re}(z) > 0$$

(*Hint:* Let $s = x + jy$; $z = u + jv$. Rationalize the fraction, and equate real and imaginary parts of $z$ and the rationalized fraction. Now consider what happens to the circle $x^2 + y^2 = 1$. To show that the *inside* of the circle goes over to the right half-plane, consider a convenient point inside the circle.)

On the basis of this transformation, deduce an extension of the Routh criterion that will determine whether the system has roots inside the unit circle. Why might this information be of interest? How can the transformation be modified to consider circles of other radii?

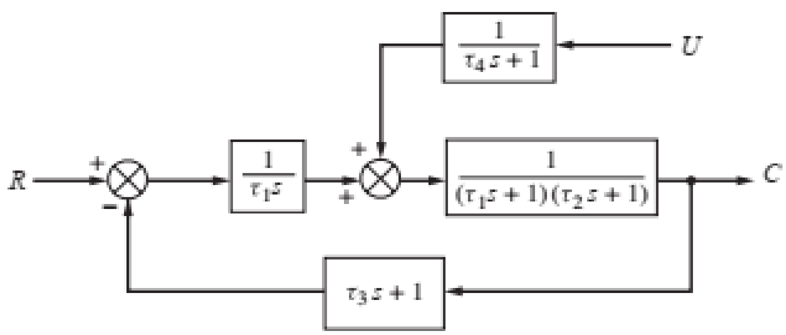



**FIGURE P13–8**

**13.8.** Given the control diagram shown in Fig. P13–8, deduce by means of the Routh criterion those values of $\tau_I$ for which the output $C$ is stable for all inputs $R$ and $U$.

**13.9.** In the control system shown in Fig. P13–9, find the value of $K_c$ for which the system is on the verge of instability. The controller is replaced by a PD controller, for which the transfer function is $K_c(\tau_D s + 1)$. If $K_c = 10$, determine the range of $\tau_D$ for which the system is stable.

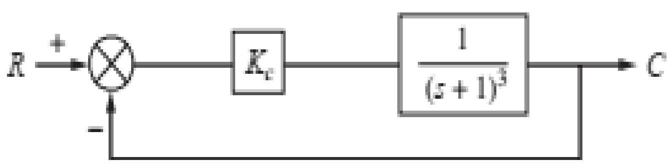



**FIGURE P13–9**

**13.10.** (*a*) Write the characteristic equation for the control system shown in Fig. P13–10.
(*b*) Use the Routh test to determine if the system is stable for $K_c = 4$.
(c) Determine the ultimate value of $K_c$ above which the system is unstable.

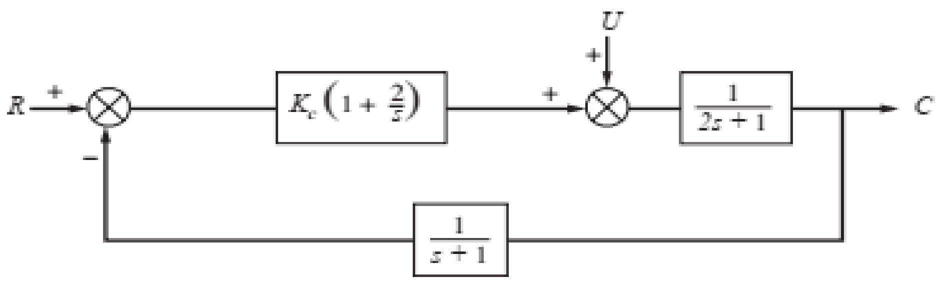



**FIGURE P13–10**

**13.11.** For the control system in Fig. P13–11, the characteristic equation is

$$s^4 + 4s^3 + 6s^2 + 4s + (1 + K) = 0$$

(*a*) Determine the value of $K$ above which the system is unstable.
(b) Determine the value of $K$ for which two of the roots are on the imaginary axis, and determine the values of these imaginary roots and the remaining two roots.

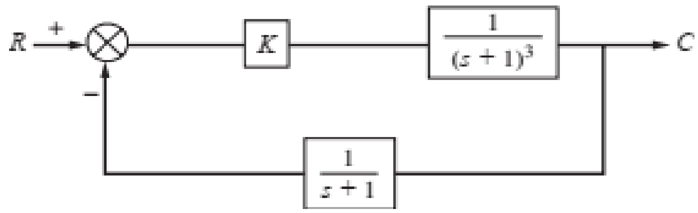



**FIGURE P13–11**