Chapter One: Review of Laplace Transforms

Chapters Two and Three in the textbook

Introduction

Laplace transform techniques provide powerful tools in numerous fields of technology such as Control Theory where knowledge of the system transfer function is essential and where the Laplace transform comes into its own.

Definition

The Laplace transform of an expression $f(t)$ is denoted by $L\{f(t)\}$ and is defined as the semi-infinite integral:

$$L\{f(t)\} = \int_{t=0}^{\infty} f(t)e^{-st}\,dt .$$

The parameter *s* is assumed to be positive and large enough to ensure that the integral converge. In more advanced applications *s* may be complex and in such cases the real part of *s* must be positive and large enough to ensure convergence.

In determining the transform of an expression, you will appreciate that the limits of the integral are substituted for *t*, so that the result will be an expression in *s*. Therefore:

$$\boxed{L\{f(t)\} = \int_{t=0}^{\infty} f(t)e^{-st}\,dt = F(s)}$$

Simple Transforms

Example: Find the Laplace transform of $f(t)=1$

Solution:

$$L\{1\}=\int_0^\infty 1e^{-st}dt=-\left[\frac{e^{-st}}{s}\right]_0^\infty=-(\frac{e^{-s\infty}}{s}-\frac{e^{-s0}}{s})=-(\frac{0}{s}-\frac{1}{s})=\frac{1}{s}$$

Example: Find the Laplace transform of $f(t)=a$, where a is a constant.

Solution:
$$L\{a\}=\int_0^\infty ae^{-st}dt=a\left[\frac{-e^{-st}}{s}\right]_0^\infty=a(\frac{-e^{-s\infty}}{s}+\frac{e^{-s0}}{s})=a(\frac{0}{s}+\frac{1}{s})=\frac{a}{s}$$

Example: Find the Laplace Transform of $f(t)=t$

Solution:

$$\mathrm{L}\{t\}=\int_0^\infty te^{-st}dt$$

Use integration by parts with

$$\int_0^\infty u\cdot dv=u\cdot v\Big|_0^\infty-\int_0^\infty v\cdot du$$

$$u=t\Rightarrow du=dt$$

$$dv=e^{-st}dt\Rightarrow v=\frac{-e^{-st}}{s}$$

$$L[t]=\int_0^\infty te^{-st}dt=-t\cdot\frac{1}{s}e^{-st}\Big|_0^\infty-\int_0^\infty\left(-\frac{1}{s}e^{-st}\right)dt=t\cdot\frac{1}{s}e^{-st}\Big|_0^\infty+\frac{1}{s}\int_0^\infty e^{-st}dt$$

$$=(0-0)+\frac{1}{s}\left[-\frac{1}{s}e^{-st}\right]_0^\infty=\frac{e^{-st}}{s^2}=\frac{1}{s^2}(1-0)=\frac{1}{s^2}$$

$$\Rightarrow L\{t\}=\frac{1}{s^2}$$

In practice we do not usually need to integrate to find Laplace transforms, instead we use a table, which allow us to read off most of the transforms we need.

Function $f(t)$	**Transform** $F(s)$	**Valid for …**
1	$\frac{1}{s}$	$s>0$
a	$\frac{a}{s}$	$s>0$
t	$\frac{1}{s^2}$	$s>0$
t^n	$\frac{n!}{s^{n+1}}$	n = positive integer
$\sin at$	$\frac{a}{s^2+a^2}$	
$\cos at$	$\frac{s}{s^2+a^2}$	
$\sinh at$	$\frac{a}{s^2-a^2}$	
$\cosh at$	$\frac{s}{s^2-a^2}$	
e^{-at}	$\frac{1}{s+a}$	
te^{-at}	$\frac{1}{(s+a)^2}$	
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$	
$e^{-at}\sin wt$	$\frac{w}{(s+a)^2+w^2}$	
$e^{-at}\cos wt$	$\frac{s+a}{(s+a)^2+w^2}$	

Rules of Laplace Transform

The Laplace transform is a linear transform by which is meant that:

1. The transform of a sum (or difference) of expressions is the sum (or difference) of the individual transforms. That is

$$\boxed{L\{f(t) \pm g(t)\} = L\{f(t)\} \pm L\{g(t)\}}.$$

2. The transform of an expression that is multiplied by a constant is the constant multiplied by the transform of the expression. That is

$$\boxed{L\{kf(t)\} = kL\{f(t)\}}.$$

Example: Determine the Laplace transform of $2e^{-t} + t$.

Solution:

$$L\{2e^{-t} + t\} = 2L\{e^{-t}\} + L\{t\} = \frac{2}{s+1} + \frac{1}{s^2} = \frac{2s^2 + (s+1)}{s^2(s+1)} = \frac{2s^2 + s + 1}{(s^3 + s^2)}$$

Example: Determine the Laplace transform of $3t^3 + \sin t$.

Solution: $$L\{3t^3 + \sin t\} = 3L\{t^3\} + L\{\sin t\} = 3 \times \frac{3!}{s^{3+1}} + \frac{1}{s^2 + 1^2} = \frac{18}{s^4} + \frac{1}{s^2 + 1}$$

$$= \frac{18(s^2+1) + 1(s^4)}{s^4(s^2+1)} = \frac{18(s^2+1) + s^4}{s^4(s^2+1)} = \frac{18s^2 + 18 + s^4}{s^4(s^2+1)}$$

Theorems of Laplace Transform

There are three important and useful theorems that enable us to deal with rather more complicated expressions.

<u>Theorem 1: First shift theorem</u>

The first shift theorem states that if $L\{f(t)\} = F(s)$ then $\boxed{L\{e^{-at} f(t)\} = F(s+a)}$

$$L\{e^{-at}f(t)\}=\int_{t=0}^{\infty}e^{-at}f(t)e^{-st}dt=\int_{t=0}^{\infty}f(t)e^{-(s+a)t}dt=F(s+a)$$

We know that $L\{e^{-at}f(t)\}=F(s+a)$ and we know that $L\{f(t)\}=F(s)$ therefore the transform $L\{e^{-at}f(t)\}$ is thus the same as $L\{f(t)\}$ with s everywhere in the result replaced by $(s+a)$.

Example: find $L\{e^{-3t}\sin 2t\}$.

Solution: We know that $L\{e^{-at}\}=\dfrac{1}{s+a}$ and $L\{\sin 2t\}=\dfrac{2}{s^2+4}$ We have $a=3$, therefore

$$L\{e^{-3t}\sin 2t\}=\frac{2}{(s+3)^2+4}=\frac{2}{s^2+3s+3s+9+4}=\frac{2}{s^2+6s+13}$$

Example: Determine the Laplace transform of $e^{3t}(t^2+4)$.

Solution: We know that $L\{t^2+4\}=\dfrac{2}{s^3}+\dfrac{4}{s}$ also that $L\{e^{3t}\}=\dfrac{1}{s-3}$ Therefore

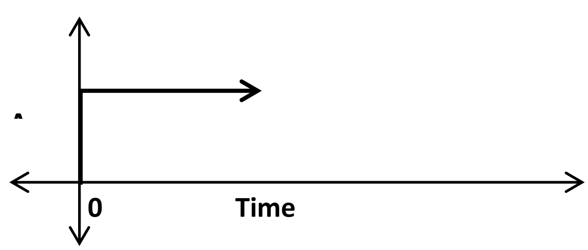
$$L\{e^{3t}(t^2+4)\}=\frac{2}{(s-3)^3}+\frac{4}{(s-3)}=\frac{2+4(s-3)^2}{(s-3)^3}=\frac{2+4(s-3)(s-3)}{(s-3)^3}$$
$$=\frac{2+4(s^2-6s+9)}{(s-3)^3}=\frac{2+4s^2-24s+36}{(s-3)^3}=\frac{4s^2-24s+38}{(s-3)^3}$$

Special Laplace Transform Functions

1- Step function

$$f(t)=\begin{cases}0 & t<0\\ A & t\geq 0\end{cases}$$

$$f(s)=\frac{A}{s}$$



If A=1 the change is called **unit step change**

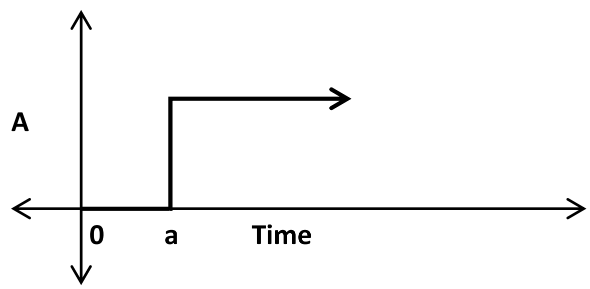
$$f(t)=\begin{cases}0 & t<0\\ 1 & t\geq 0\end{cases}$$

$$f(s)=\frac{1}{s}$$

Step function with Time Delay

$$f(t)=\begin{cases}0 & t<a\\ A & t\geq a\end{cases}$$

$$f(s)=\frac{A}{s}e^{-as}$$

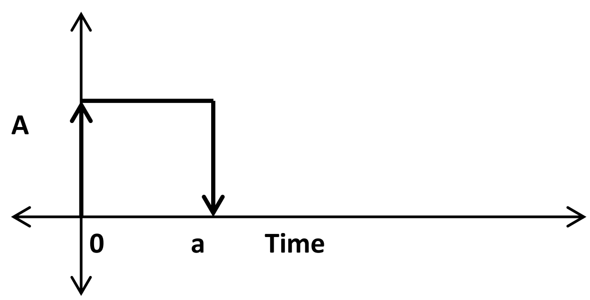


2. Pulse function

$$f(t)=\begin{cases}0 & t<0\\ A & 0\leq t\leq a\\ 0 & t\geq a\end{cases}$$

$$f(s)=\frac{A}{s}-\frac{A}{s}e^{-as}$$

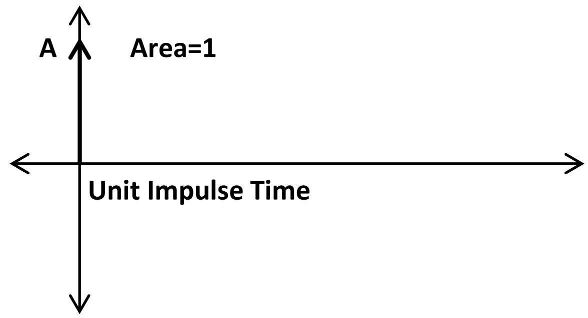
$$=\frac{A}{s}(1-e^{-as})$$



$$L\{f(t)\}=\int_0^a Ae^{-st}+\int_a^\infty 0e^{-st}=-\frac{At}{s}e^{-st}\Big|_0^a=-\frac{A}{s}(e^{-sa}-e^{-s0})=\frac{A}{s}(1-e^{-sa})$$

if A=1➔unit Pulse (Impulse)

3. Impulse function



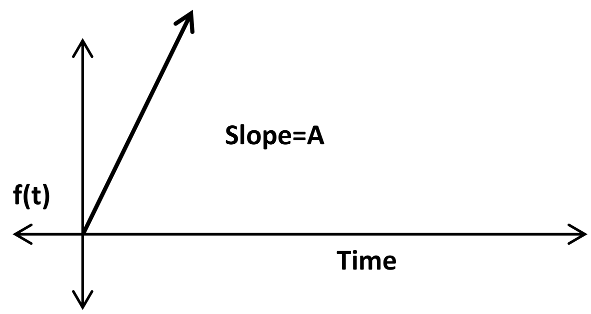
$$f(t) = \begin{cases} 0 & t < 0 \\ A & 0 \le t \le \delta t \\ 0 & t \ge \delta t \end{cases}$$

$$f(s) = area = A \times \delta t$$

This function is represented by δ(t). The *unit impulse* function is a special case of the pulse function with zero width (tw →0) and unit pulse area (so a = 1/tw). Taking the limit:

$$L\{\delta(t)\} = \lim_{t_w \to 0} \frac{1}{t_w s}[1 - e^{-st_w}] = \lim_{t_w \to 0} \frac{1}{s}[-e^{-st_w}] = 1$$

4. Ramp function



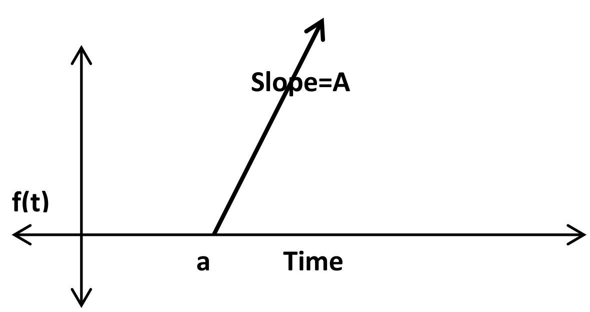
$$f(t) = \begin{cases} 0 & t < 0 \\ At & t \ge 0 \end{cases}$$

$$f(s) = \frac{A}{s^2}$$

$$L\{At\} = \int_0^\infty Ate^{-st} = -\frac{At}{s}e^{-st}\Big|_0^\infty + \int \frac{A}{s}e^{-st}dt = -\frac{At}{s}e^{-st}\Big|_0^\infty + \frac{A}{s}\frac{-1}{s}e^{-st}\Big|_0^\infty$$

$$= \frac{-A}{s}(\infty e^{-\infty} - 0e^{-0}) - \frac{A}{s^2}(e^{-\infty} - e^{-0}) = \frac{A}{s^2}$$

Ramp function with time delay



$$f(t) = \begin{cases} 0 & t < a \\ At & t \ge a \end{cases}$$

$$f(s) = \frac{A}{s^2}e^{-as}$$

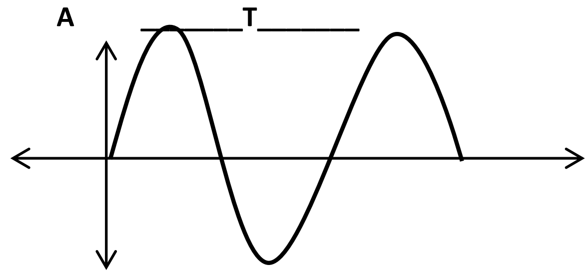
5. Sine function

$$f(t) = \begin{cases} 0 & t < a \\ A\sin\ wt & t \geq a \end{cases}$$

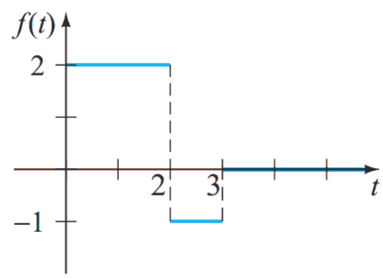
$$f(s) = \frac{A\omega}{s^2 + \omega^2}$$

$$\omega = 2\pi f$$

$$T = \frac{1}{f}$$



Example: Find the Laplace transform for



Solution:

1. At t=0 the function looks like the very basic unit step function. But unit function knows only about 0 and 1, here we have f(t)=2. That means we have to use 2u(t).

2. Then in time t=2 its value changes from 2 to −1 (i.e. 3 down at t=2) which means we have to add −3u(t−2).

3. Finally the value at t=3 jumps 1 higher, which brings member u(t−3).

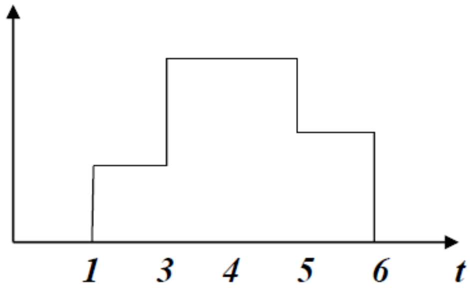
f(t)=2u(t)−3u(t−2)+u(t−3)

So far we collected unit step functions to express function from the graph.

$$L\{f(t)\} = L\{2\text{u(t)} - 3\text{u(t - 2)} + \text{u(t - 3)}\} = \text{L}\{2\text{u(t)}\} - 3\text{L}\{\text{u(t - 2)}\} + \text{L}\{\text{u(t - 3)}\} = \frac{2}{s} - \frac{3}{s}e^{-2s} + \frac{1}{s}e^{-3s}$$

Example: Determine the Laplace transform of the function

$$f(t)=\begin{cases}0 & t<1\\ 1 & 1\le t\le 3\\ 3 & 3\le t\le 5\\ 2 & 5\le t\le 6\\ 0 & t\ge 6\end{cases}$$

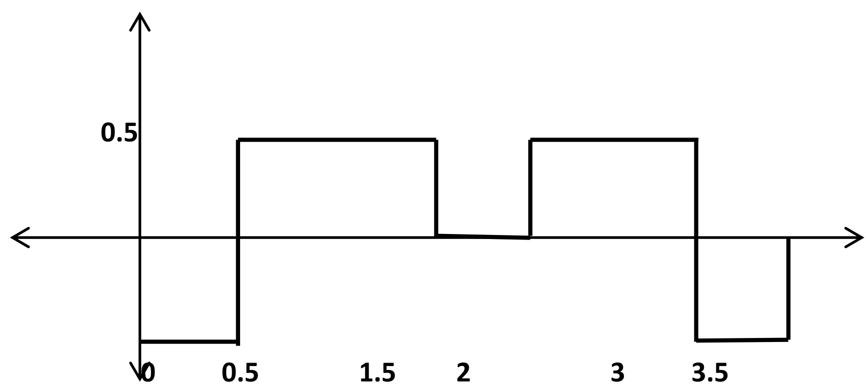


Solution:

$$f(t)=0u(t-0)+1u(t-1)+2u(t-3)-1u(t-5)-2u(t-6)$$

$$F(s)=\frac{1}{s}e^{-s}+\frac{2}{s}e^{-3s}-\frac{1}{s}e^{-5s}-\frac{2}{s}e^{-6s}=\frac{1}{s}(e^{-s}+2e^{-3s}-e^{-5s}-2e^{-6s})$$

<u>Example:</u>



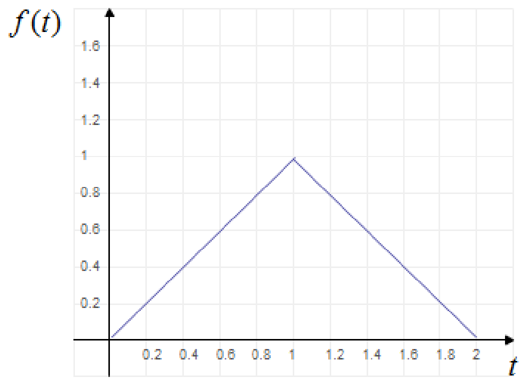
Solution:

$$f(t)=-0.5+1u(t-0.5)-0.5u(t-1.5)+0.5u(t-2)-1u(t-3)+0.5u(t-3.5)$$

$$f(s)=-\frac{0.5}{s}+\frac{1}{s}e^{-0.5s}-\frac{0.5}{s}e^{-1.5s}+\frac{0.5}{s}e^{-2s}-\frac{1}{s}e^{-3s}+\frac{0.5}{s}e^{-3.5s}$$

Example: Find F(s) for

$$f(t)=\begin{cases}0 & t<0\\ t & 0<t<1\\ 2-t & 1<t<2\\ 0 & t>2\end{cases}$$

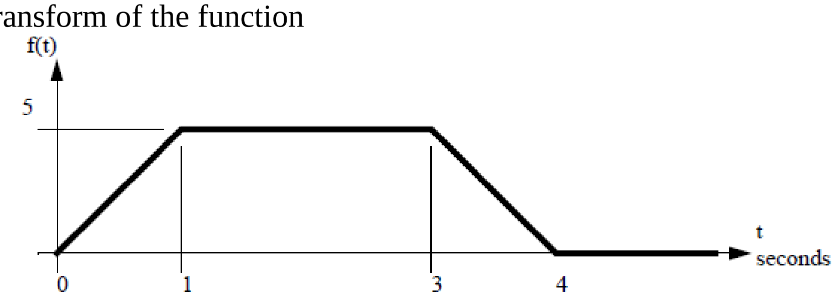


Solution:

$$f(t)=tU(t)-t(t-1)U(t-1)-(t-1)U(t-1)+(t-2)U(t-2)$$
$$=tU(t)-2(t-1)U(t-1)+(t-2)U(t-2)$$

$$L[f(t)]=L[tU(t)-2(t-1)U(t-1)+(t-2)U(t-2)]$$
$$=L[tU(t)]-L[2(t-1)U(t-1)]+L[(t-2)U(t-2)]$$
$$=\frac{1}{s^2}-\frac{2}{s^2}e^{-s}+\frac{1}{s^2}e^{-2s}$$

Example: Determine the Laplace transform of the function



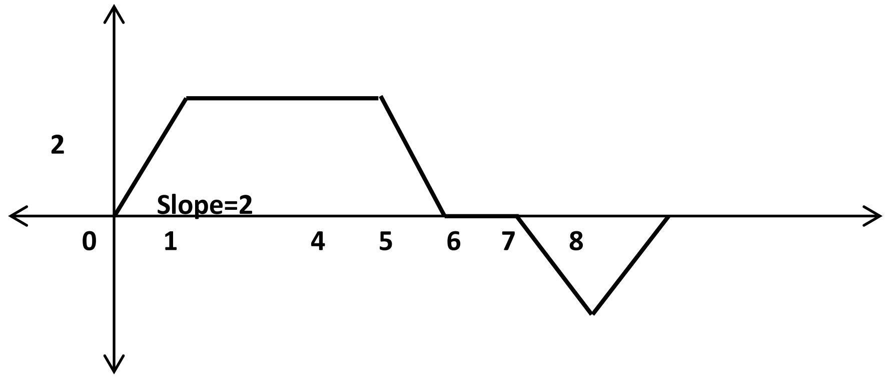
Solution:

$$f(t)=5tu(t)-5(t-1)u(t-1)-5(t-3)u(t-3)+5(t-4)u(t-4)$$

$$F(s)=\frac{5}{s^2}-\frac{5}{s^2}e^{-s}-\frac{5}{s^2}e^{-3s}+\frac{5}{s^2}e^{-4s}=\frac{5}{s^2}(1-e^{-s}-e^{-3s}+e^{-4s})$$

Example: Find the Laplace transform of f(t) shown in Fig.

$$f(t)=\begin{cases} 2t & 0\le t<1 \\ 2 & 1\le t<4 \\ -2t & 4\le t<5 \\ 0 & 5\le t<6 \\ -2t & 6\le t<7 \\ 2t & 7\le t<8 \\ 0 & t>8 \end{cases}$$



Solution:

$$f(t)=2tu(t)-2(t-1)u(t-1)-2(t-4)u(t-4)+2(t-5)u(t-5)-2(t-6)u(t-6)$$
$$+2\times 2(t-7)u(t-7)-2(t-8)u(t-8)$$
$$f(s)=\frac{2}{s^2}-\frac{2}{s^2}e^{-s}-\frac{2}{s^2}e^{-4s}+\frac{2}{s^2}e^{-5s}-\frac{2}{s^2}e^{-6s}+\frac{4}{s^2}e^{-7s}-\frac{2}{s^2}e^{-8s}$$

Rational Functions Technique: Partial fraction

Often it is necessary to break down a complicated rational function of the form $\frac{P(s)}{Q(s)}$ (where *P(s)* and *Q(s)* are polynomials in *s*, and the degree of the top polynomial is less than the degree of the bottom polynomial), into the sum of simpler fractions called ***Partial Fractions***.

The type of partial fraction that you use depends on the factors of the bottom polynomial**.**

We will look at 3 cases:

Case 1: All the factors of the bottom *Q(s)* are linear and non-repeating**.**

Case 2: *Q(s)* has some repeated linear factors.

Case 3: *Q(s)* has some irreducible quadratic factors.

Examples of case 1:

$\frac{s}{(s+2)(s-1)}$ or $\frac{3+2s}{s(s+5)(s+3)}$

Examples of case 2:

$\frac{2s+1}{(s-8)^2}$ or $\frac{2}{s(s-3)^2}$ or $\frac{s-1}{(s+1)^2(s+2)}$ or $\frac{s^2-1}{(s+3)^2(s-1)^3}$

Examples of case 3:

$\frac{2s+1}{s(s^2+4)}$ or $\frac{3}{((s-3)^2+1)(s+2)}$ or $\frac{4s^2+3s-1}{(s-1)(s^2+3)^2((s+1)^2+9)^2}$

Case 1: All the factors of the bottom *Q(s)* are linear and non-repeating.

Case 1: All the bottom factors are linear (i.e. of the form x $\pm$ some number) and then is no repeated factor (i.e. there is no factor which is squared or cubed, etc.) and there are no irreducible quadratic terms (don't worry about this!). In this case therefore, we are talking about rational functions of the form: $\frac{top\,polynomial}{(x-a)(x-b)...(x-g)}$

In this case we can rewrite the rational function as follows:

$$\boxed{\frac{top\,polynomial}{(x-a)(x-b)...(x-g)} = \frac{A}{x-a} + \frac{B}{x-b} + ... + \frac{G}{x-g}}$$

Example

Show that $\frac{1}{(s+7)(s+3)} = \frac{-\frac{1}{4}}{s+7} + \frac{\frac{1}{4}}{s+3}$

Solution

This is a case 1 partial fraction so we start with

$$\frac{1}{(s+7)(s+3)} = \frac{A}{s+7} + \frac{B}{s+3}$$

Step1: Remove Fractions

Multiply both sides of the equation by the denominator on the left hand side

$$\frac{1\times(s+7)(s+3)}{(s+7)(s+3)}=\frac{A(s+7)(s+3)}{s+7}+\frac{B(s+7)(s+3)}{s+3}$$
$$\Rightarrow 1=A(s+3)+B(s+7)$$

Step2: Choose s values to find A and B

The equation above is true for ***all*** values of s. We can choose s values to make things simple:

Choose s = -3 so that (s+3) = 0 and we have

$$1=A(0)+B(4)\ \Rightarrow\ 1=4B\ \Rightarrow\ B=\frac{1}{4}$$

Choose s = -7 so that (s+7) = 0 and we have

$$1=A(-4)+B(0)\ \Rightarrow\ 1=-4A\ \Rightarrow\ A=-\frac{1}{4}$$

Step3: Substitute A and B into the original expression

$$\frac{1}{(s+7)(s+3)}=\frac{-\frac{1}{4}}{s+7}+\frac{\frac{1}{4}}{s+3}$$

Example: Write $\frac{3s-1}{(s+1)(s+2)}$ in partial fraction form

Solution:

write $\frac{3s-1}{(s+1)(s+2)}=\frac{A}{s+1}+\frac{B}{s+2}$

Multiply both sides by $(s+1)(s+2)$:

$$\frac{3s-1}{(s+1)(s+2)}(s+1)(s+2)=\frac{A}{s+1}(s+1)(s+2)+\frac{B}{s+2}(s+1)(s+2)$$

$$\Rightarrow 3s-1=A(s+2)+B(s+1)$$

If choose s = -2

$$\Rightarrow 3(-2)-1=A(-2+2)+B(-2+1) \Rightarrow -7=A.0+B(-1) \Rightarrow -7=-B$$

$$\Rightarrow B=7$$

If choose s = -1 $\Rightarrow 3(-1)-1=A(-1+2)+B(-1+1) \Rightarrow -4=A(1)+B(0) \Rightarrow -4=A$

$$\Rightarrow A=-4$$

Therefore $\dfrac{3s-1}{(s+1)(s+2)}=\dfrac{-4}{s+1}+\dfrac{7}{s+2}$

Example

Find the fixed constants A, B, C so that the partial fraction decomposition can be completed:

$$\frac{4s+3}{s(s-1)(s+3)}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s+3}$$

Solution:

$$\frac{4s+3}{s(s-1)(s+3)}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s+3}$$

Multiplying both sides of this equation by the left hand side denominator we deduce that,

$$s(s-1)(s+3)\frac{4s+3}{s(s-1)(s+3)}=s(s-1)(s+3)\frac{A}{s}+s(s-1)(s+3)\frac{B}{s-1}+s(s-1)(s+3)\frac{C}{s+3}$$

$$\Rightarrow 4s+3=A(s-1)(s+3)+Bs(s+3)+Cs(s-1)$$

Now in turn, put, $s=0$, $s=1$, $s=-3$. You will find at each stage that **<u>2</u>** of the A, B, C terms will vanish:

If choose s = 0

$$4(0)+3 = A(0-1)(0+3)+B(0)+C(0) \Rightarrow 3 = -3A \Rightarrow A = 1$$

If choose s = 1

$$4(1)+3 = A(0)+B(1)(1+3)+C(0) \Rightarrow 7 = 4B \Rightarrow B = \frac{7}{4}$$

If choose s = -3

$$4(-3)+3 = A(0)+B(0)+C(-3)(-3-1) \Rightarrow -9 = 12C \Rightarrow B = \frac{-9}{12}$$

Therefore:

$$\frac{4s+3}{s(s-1)(s+3)} = \frac{[-1]}{s} + \frac{\left[\frac{7}{4}\right]}{s-1} + \frac{\left[\frac{-3}{4}\right]}{s+3} = -\frac{1}{s} + \frac{7}{4(s-1)} - \frac{3}{4(s+3)}$$

Case 2: Denominator *Q(s)* has some repeated linear factors.

Sometimes a linear factor is repeated twice or three times or four times, etc. This means that we will have an expression like $(s - \text{some number})^2$ or $(s - \text{some number})^3$ or ... below the line. When this occurs, you have to be careful as we have to use a partial fraction for each of the powers. If you want, you can use the following table:

Factor in given rational function	**Corresponding partial fraction**
$\frac{top\ polynomial}{s \pm some\ number}$	$\frac{A}{s \pm some\ number}$
$\frac{top\ polynomial}{(s \pm somenumber)^2}$	$\frac{A}{s \pm some\ number} + \frac{B}{(s \pm some\ number)^2}$
$\frac{top\ polynomial}{(s \pm some\ number)^3}$	$\frac{A}{s \pm some\ number} + \frac{B}{(s \pm some\ number)^2} + \frac{C}{(s \pm some\ number)^3}$

Examples of case 2:

$$\frac{4s^2+3s-2}{(s-1)^2(s+3)}=\frac{A}{(s-1)^2}+\frac{B}{s-1}+\frac{C}{s+3}\ , \quad A,B,C=\text{constants}$$

$$\frac{3s^4-7s^3+2s^2-5}{s^3(s+1)^2(s+5)}=\frac{A}{s^3}+\frac{B}{s^2}+\frac{C}{s}+\frac{D}{(s+1)^2}+\frac{E}{(s+1)}+\frac{F}{(s+5)} \quad A,B,C,D,E,F=\text{constants}$$

Example:

Write in terms of partial fractions $\frac{8s-1}{(s-1)^2}$

Solution:

In this case $\frac{8s-1}{(s-1)^2}=\frac{A}{s-1}+\frac{B}{(s-1)^2}$

Step1 we multiply both sides by $(s-1)^2$ to remove fractions

$$\frac{8s-1}{(s-1)^2}(s-1)^2=\frac{A}{s-1}(s-1)^2+\frac{B}{(s-1)^2}(s-1)^2 \Rightarrow 8s-1=A(s-1)+B$$

Step2: we can choose *s* = 1 as before so that (s-1) = 0 and we get

$$8-1=B \ \Rightarrow \ B=7$$

We cannot choose another s value to directly find A however. There is more than one approach to finding A but the easiest method is called "***equating coefficients***". In this case, we note that there must be the same "amount of s" on both sides of the equation. On the left hand side, we have 8s and on the right we have As, so that A must be 8.

Step3; write the answer down

$$\frac{8s-1}{(s-1)^2}=\frac{8}{s-1}+\frac{7}{(s-1)^2}$$

Example

Write $\frac{2s+1}{(s-1)(s+2)^2}$ in partial fraction form.

Solution:

As the linear factor (s+2) repeats (i.e. we have (s+2) and $(s+2)^2$ in the denominator) the required partial fraction is of the form,

$$\frac{2s+1}{(s-1)(s+2)^2}=\frac{A}{s-1}+\frac{B}{s+2}+\frac{C}{(s+2)^2}$$

where A, B and C are fixed constants which are to be found.

Step1: multiply both sides by the denominator $(s - 1)(s+2)^2$

$$(s-1)(s+2)^2\frac{2s+1}{(s-1)(s+2)^2}=(s-1)(s+2)^2\frac{A}{s-1}+(s-1)(s+2)^2\frac{B}{s+2}+(s-1)(s+2)^2\frac{C}{(s+2)^2}$$

$$\Rightarrow 2s+1=(s+2)^2A+(s-1)(s+2)B+(s-1)C$$

The latter equation holds for **all** values of the variable s.

Step2: Choose good s values

Choose s = -2 $\Rightarrow 2(-2)+1=(0)^2A+(0)B+(-2-1)C \Rightarrow -3=-3C \;\Rightarrow\; C=1$

Choose s = 1 $\Rightarrow 2(1)+1=(1+2)^2A+(0)B+(0)C \Rightarrow 3=9A \;\Rightarrow\; A=\frac{1}{3}$

We again have the problem of not being able to choose a good s value to find B. We again "equate coefficients". The best strategy is to equate the highest power of s on both sides, which is s^2 . On the left hand side of (1) we have 0 lots of s^2. On the right hand side of (1) the $(s+2)^2 A$ term will contribute As^2 if you multiply it out and the $(s-1)(s+2)B$ term will contribute Bs^2. So we have

$$0 = A + B \qquad \Rightarrow B = -A = -\frac{1}{3}$$

This method can involve more calculation though. Either way

$$\frac{2s+1}{(s-1)(s+2)^2} = \frac{1}{3}.\frac{1}{s-1} - \frac{1}{3}.\frac{B}{s+2} + \frac{1}{(s+2)^2}$$

Case 3 Denominator *Q(s)* has some irreducible quadratic factors.

Irreducible means that the quadratic term in the denominator cannot be factorized into two brackets.

Examples of case 3:

$$\frac{3s^2-2s-5}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} \quad , \quad A, B, C = \text{constants}$$

$$\frac{s^3-s+2}{s(s-1)(s^2+2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{Cs+D}{s^2+2} \quad , \quad A, B, C, D = \text{constants}$$

Example:

Write $\dfrac{5s^2-7s+8}{(s-1)(s^2-2s+5)}$ in partial fractions.

Solution

The denominator contains an irreducible quadratic term (i.e, it cannot be easily factored into two linear terms. If it did factor into linear factors, then we would be back to cases 1 or 2.) As it does not in this example we must write:

$$\frac{5s^2-7s+8}{(s-1)(s^2-2s+5)}=\frac{A}{(s-1)}+\frac{Bs+C}{(s^2-2s+5)}$$

Note: Linear term on bottom means ***constant*** A on top. Quadratic term on bottom means ***linear*** term *Bs* + *C* on top.

As before to find these constants multiply both sides by the denominator

(s – 1) (s^2-2s+5).

$$(s-1)(s^2-2s+5)\frac{5s^2-7s+8}{(s-1)(s^2-2s+5)}=(s-1)(s^2-2s+5)\frac{A}{(s-1)}+(s-1)(s^2-2s+5)\frac{Bs+C}{(s^2-2s+5)}$$

Which gives: $5s^2-7s+8=(s^2-2s+5)A+(s-1)(Bs+C)$

The latter equation holds for **all** values of the variable *s*. So we can choose any values for *s* and set up three simultaneous equations for *A*, *B* and *C*.

By putting *s* = 1 we can get one value easily:

Choose s = 1

$$\Rightarrow 5(1)^2-7(1)+8=(1^2-2(1)+5)A+0(B(1)+C)\Rightarrow 6=4A+0\Rightarrow A=\frac{3}{2}$$

We cannot make any more brackets = 0 by a good choice of s. As for case 2 however, we can equate coefficients. Start with the highest power s^2 first

Equate s^2 $\Rightarrow 5=A+B \Rightarrow B=5-A \Rightarrow B=5-\frac{3}{2}=\frac{7}{2}$

Equate s $\Rightarrow -7=-2A-B+C\Rightarrow -7=-2(\frac{3}{2})-\frac{7}{2}+C \Rightarrow -7=-\frac{13}{2}+C$

$\Rightarrow C=-\frac{1}{2}$

Inverse Laplace Transforms

The Laplace transform is an expression in the variable s which denoted by $F(s)$. It is said that $f(t)$ and $F(s) = L\{f(t)\}$ form a transform pair. This means that if $F(s)$ is the Laplace transform of $f(t)$ then $f(t)$ is the inverse Laplace transform of $F(s)$. We write as:

$$f(t) = L^{-1}\{F(s)\} \quad \text{or} \quad L^{-1}\{F(s)\} = f(t)$$

The operator L^{-1} is known as the operator for inverse Laplace transform. There is no simple integral definition of the inverse transform so you have to find it by working backwords.

Here we have the reverse process, i.e. given a Laplace transform, we have to find the function of t to which it belongs. We use the following table:

Table of inverse transforms

$F(s)$	$f(t)$
$\frac{a}{s}$	a
$\frac{1}{s+a}$	e^{-at}
$\frac{n!}{s^{n+1}}$	t^n
$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$
$\frac{a}{s^2+a^2}$	$\sin at$
$\frac{s}{s^2+a^2}$	$\cos at$
$\frac{a}{s^2-a^2}$	$\sinh at$
$\frac{s}{s^2-a^2}$	$\cosh at$

Two Properties of Laplace Transform Inverse

Both Laplace transform and its inverse are linear transforms, by which is meant that:

i. The transform of a sum (or difference) of expressions is the sum (or difference of the individual transforms. That is:

$$L^{-1}\{F(s) \pm G(t)\} = L^{-1}\{F(s)\} \pm L^{-1}\{G(s)\}$$

ii. The transform of an expression that is multiplied by a constant is the constant multiplied by the transform of the expression. That is:

$$L^{-1}\{kF(s)\} = kL^{-1}\{F(s)\}$$ where k is constant

Example: find $L^{-1}\left\{\frac{1}{s-2}\right\}$?

Solution: $L^{-1}\left\{\frac{1}{s-2}\right\} = L^{-1}\left\{\frac{1}{s+(-2)}\right\} = e^{2t}$.

Example: find $L^{-1}\left\{\frac{8}{s^2+64}\right\}$?

Solution: We can write the inverse transform as we know that $L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$.

Here we have $a = 8$ therefore: $L^{-1}\left\{\frac{8}{s^2+64}\right\} = L^{-1}\left\{\frac{8}{s^2+8^2}\right\} = \sin 8t.$

Example: find $L^{-1}\left\{\frac{12}{s^2-9}\right\}$?

Solution: $L^{-1}\left\{\frac{12}{s^2-9}\right\} = 4L^{-1}\left\{\frac{3}{s^2-9}\right\} = 4\sinh 3t$

Example: Find the inverse Laplace transform for $F(s) = -\frac{2}{3s-4}$.

Solution:

$$L^{-1}\left\{-\frac{2}{3s-4}\right\} = L^{-1}\left\{-\frac{2}{3}\left(\frac{1}{s-\frac{4}{3}}\right)\right\} = -\frac{2}{3}L^{-1}\left\{\frac{1}{s-\frac{4}{3}}\right\} = -\frac{2}{3}e^{\frac{4}{3}t}$$

Example: Determine $L^{-1}\left\{\frac{3s+1}{s^2-s-6}\right\}$.

Solution: This certainly did not appear in our list of standard transforms but if we write $\frac{3s+1}{s^2-s-6}$ as the sum of two simpler functions, i.e.

$\frac{3s+1}{s^2-s-6} = \frac{1}{s+2} + \frac{2}{s-3}$ it makes all the difference.

This is simply the method of writing the more complex algebraic fraction in terms of its partial fractions which we have previously seen.

We can now proceed to find

$$L^{-1}\left\{\frac{3s+1}{s^2-s-6}\right\} = L^{-1}\left\{\frac{1}{s+2} + \frac{2}{s-3}\right\} = e^{-2t} + 2e^{3t}$$

Example: Determine $L^{-1}\left\{\frac{9s-8}{s^2-2s}\right\}$.

Solution: Simplify: $L^{-1}\left\{\frac{9s-8}{s^2-2s}\right\} = L^{-1}\left\{\frac{9s-8}{s(s-2)}\right\}$

Remember from partial fractions we have the form:

$$\frac{9s-8}{s(s-2)} = \frac{A}{s} + \frac{B}{(s-2)} \qquad \times s(s-2)$$

$$9s-8 = A(s-2) + Bs$$

Let s=0 then $A = \frac{0-8}{0-2} = 4$

Let s=2 then $B = \frac{9\times 2-8}{2} = 5$

This gives us $\frac{9s-8}{s(s-2)} = \frac{4}{s} + \frac{5}{(s-2)}$

So we now have to find

$$L^{-1}\left\{\frac{9s-8}{s(s-2)}\right\} = L^{-1}\left\{\frac{4}{s} + \frac{5}{(s-2)}\right\} = L^{-1}\left\{\frac{4}{s}\right\} + 5L^{-1}\left\{\frac{1}{(s-2)}\right\} = 4 + 5e^{2t}$$

Example: Determine $L^{-1}\left\{\frac{13s+11}{(s-1)(s+3)}\right\}$.

Solution: Remember from partial fractions we have the form:

$\frac{13s+11}{(s-1)(s+3)} = \frac{A}{(s-1)} + \frac{B}{(s+3)}$ Multiple both sides by $(s-1)(s+3)$

$$13s+11 = (s+3)A + (s-1)B$$

Let s=1 then $A = \frac{13+11}{1+3} = \frac{24}{4} = 6$

Let s=-3 then $B = \frac{-3\times 13+11}{-3-1} = \frac{-28}{-4} = 7$

So we now have to find

$$L^{-1}\left\{\frac{13s+11}{(s-1)(s+3)}\right\} = L^{-1}\left\{\frac{6}{(s-1)} + \frac{7}{(s+3)}\right\} = 6L^{-1}\left\{\frac{1}{(s-1)}\right\} + 7L^{-1}\left\{\frac{1}{(s+3)}\right\} = 6e^{t} + 7e^{-3t}$$

Example: Find the inverse Laplace transform of $F(s) = \frac{s+4}{(s+2)(s+1)^2}$

Solution: Expand F(s) as $F(s) = \frac{s+4}{(s+2)(s+1)^2} = \frac{A}{s+2} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)}$

$$(s+4) = (s+1)^2 A + (s+2)B + (s+1)(s+2)C$$

Let s=-1 then $B = \frac{-1+4}{-1+2} = \frac{3}{1} = 3$

Let s=-2 then $A = \frac{-2+4}{(-2+1)^2} = \frac{2}{1} = 2$

equate s $\Rightarrow 1 = 2A + B + 3C \Rightarrow\ 1 = 2\times 2 + 3 + 3C \Rightarrow C = \frac{1-7}{3} = -2$

Check by cross-multiplying:

$$\frac{s+4}{(s+2)(s+1)^2} = \frac{2}{(s+2)} + \frac{3}{(s+1)^2} - \frac{2}{(s+1)}$$

$$\frac{s+4}{(s+1)(s+3)^2} = \frac{2}{s+2} + \frac{3}{(s+1)^2} + \frac{-2}{(s+1)}$$

$$s+4 = 2(s^2+2s+1) + 3(s+2) - 2(s^2+3s+2)$$

$$s^2:\quad 0 = 2-2$$
$$s^1:\quad 1 = 4+3-6$$
$$s^0:\quad 4 = 2+6-4$$

$$L^{-1}\left\{\frac{s+4}{(s+2)(s+1)^2}\right\}=L^{-1}\left\{\frac{2}{(s+2)}+\frac{3}{(s+1)^2}-\frac{2}{(s+1)}\right\}$$

$$=2L^{-1}\left\{\frac{1}{(s+2)}\right\}+3L^{-1}\left\{\frac{1}{(s+1)^2}\right\}-2L^{-1}\left\{\frac{1}{(s+1)}\right\}=2e^{-2t}+3te^{-t}-2e^{-t}$$

Example: Find the inverse Laplace transform of $F(s)=\dfrac{s+9}{s^2+6s+13}$

Solution:

$$f(t)=L^{-1}[F(s)]=L^{-1}\left[\frac{s+9}{s^2+6s+13}\right]=L^{-1}\left[\frac{s+9}{s^2+6s+3^2-3^2+13}\right]=L^{-1}\left[\frac{s+9}{(s+3)^2+2^2}\right]$$

$$=L^{-1}\left[\frac{(s+3)+6}{(s+3)^2+2^2}\right]=L^{-1}\left[\frac{s+3}{(s+3)^2+2^2}\right]+L^{-1}\left[\frac{6}{(s+3)^2+2^2}\right]=L^{-1}\left[\frac{s+3}{(s+3)^2+2^2}\right]+L^{-1}\left[3\cdot\frac{2}{(s+3)^2+2^2}\right]$$

$$=L^{-1}\left[\frac{s+3}{(s+3)^2+2^2}\right]+3L^{-1}\left[\frac{2}{(s+3)^2+2^2}\right]=e^{-3t}\cos(2t)+3e^{-3t}\sin(2t)\ =e^{-3t}[\cos(2t)+3\sin(2t)]$$

Laplace Transform of a Derivative

Before we apply Laplace transform to solve a differential equation, we need to know the Laplace transform of a derivative. Given some expression $f(t)$ with Laplace transform $L\{f(t)\}=F(s)$, the Laplace transform of the derivative $f'(t)$ is:

$$L\{f'(t)\}=\int_{t=0}^{\infty}e^{-st}f'(t)dt$$

This can be integrated by parts as follows:

$$L\{f'(t)\}=\int_{t=0}^{\infty}e^{-st}f'(t)dt \qquad \text{where } u=e^{-st} \qquad \mathrm{d}v=f'(t)$$

$$\mathrm{d}u=-se^{-st} \qquad v=\int f'(t)=f(t)$$

$$L\{f'(t)\}=\left[e^{-st}f(t)\right]_{t=0}^{\infty}+s\int_{t=0}^{\infty}e^{-st}f(t)dt=(0-f(0))+sF(s)$$

Assuming $e^{-st} f(t) \to 0$ as $t \to \infty$

That is: $\boxed{L\{f'(t)\} = sF(s) - f(0)}$

Thus, the Laplace transform of the derivative of $f(t)$ is given in terms of the Laplace transform of $f(t)$ when $t = 0$. The next properties is very important for the above formula.

In general, to solve differential equation $af'(t) + bf(t) = g(t)$ given that $f(0) = k$ where *a*, *b*, and *k* are known constants and $g(t)$ is a known expression in *t* using Laplace transform are as follows:

i. Take the Laplace transform of both sides of the differential equation.

ii. Find the expression of $F(s) = L\{f(t)\}$ in the form of an algebraic fraction

iii. Separate *F(s)* into its partial fractions.

iv. Find the inverse Laplace transform $L\{f'(t)\}$ to find the solution *f(t)* to the differential equation.

Example: Solve $f'(t) - f(t) = 2$ where $f(0) = 0$

Solution: Taking Laplace transforms of both sides of the equation gives:

$$sF(s) - f(0) - F(s) = \frac{2}{s}$$

$$F(s)(s-1) = \frac{2}{s}$$

$$F(s) = \frac{2}{s(s-1)}$$ solve using partial fraction

$$\frac{2}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1}$$ solve for A and B where $A = -2$ and $B = 2$

$$F(s) = -\frac{2}{s} + \frac{2}{s-1}$$

The inverse transformation gives the solution as

$$f(t) = -2 + 2e^{t} = -2\left(1 - e^{t}\right)$$

Example: Solve $f'(t) - f(t) = e^{2t}$ where $f(0) = 1$

Solution:

$$sF(s) - f(0) - F(s) = \frac{1}{s-2}$$

$$F(s)(s-1) - 1 = \frac{1}{s-2}$$

$$F(s) = \frac{1}{(s-2)(s-1)} + \frac{1}{s-1} = \frac{(s-2)+1}{(s-2)(s-1)} = \frac{(s-1)}{(s-2)(s-1)} = \frac{1}{s-2}$$

The inverse transform then gives the solution as

$$f(t) = e^{2t}$$

Example: Solve $3f'(t) - 2f(t) = 4e^{-t} + 2$ where $f(0) = 0$

Solution:

$$3\left[sF(s) - f(0)\right] - 2F(s) = \frac{4}{s+1} + \frac{2}{s}$$

$$3sF(s) - 3f(0) - 2F(s) = \frac{6s+2}{s(s+1)}$$

$$F(s)(3s-2) = \frac{6s+2}{s(s+1)}$$

$F(s) = \frac{6s+2}{s(s+1)(3s-2)}$ solve using partial fractions for $A = -1$, $B = -\frac{4}{5}$, and $C = \frac{27}{5}$

$$F(s)=-\frac{1}{s}-\frac{4}{5}\left(\frac{1}{s+1}\right)+\frac{27}{15}\left(\frac{1}{3s-2}\right)=-\frac{1}{s}-\frac{4}{5}\left(\frac{1}{s+1}\right)+\frac{81}{15}\left(\frac{1}{s-\frac{2}{3}}\right)$$

The inverse transform then gives the solution as :

$$f(t)=-1-\frac{4}{5}e^{-t}+\frac{81}{15}e^{\frac{2}{3}t}$$

1.9 Laplace Transform of integers

If f(t) is a function having Laplace transform F(s)= L{f(t)}, then the Laplace transform of the integration of a function f(t), is given by:

$$L\int_0^t f(t)dt=\frac{F(s)}{s}$$

Proof:

$$L\int_0^t f(t)dt=\int_0^\infty\left(\int_0^t f(t)\right)e^{-st}dt=\left[\frac{-1}{s}e^{-st}\int_0^t f(t)\right]_0^\infty+\frac{1}{s}\int_0^t f(t)e^{-st}dt=\frac{1}{s}F(s)$$

Initial value theorem

It can be used to find the steady-state value of a system (providing that a steady-state value exists).

if $LF(t)=F(s)$, then

$$F(0)=\lim_{t\to 0}F(t)=\lim_{s\to\infty}sF(s)$$

1.12 Final value theorem

$$\lim_{t\to\infty}F(t)=\lim_{s\to 0}sF(s)$$

Example:

Find the final value of the function x(t) for which the laplace inverse is: -

$$x(s)=\frac{1}{s(s^3+3s^2+3s+1)}$$

$$\lim_{t\to\infty} x(t)=\lim_{s\to 0} sx(s)=\lim_{s\to 0}\frac{s\times 1}{s(s^3+3s^2+3s+1)}$$

$$=\lim_{s\to 0}\frac{1}{(s^3+3s^2+3s+1)}=1$$

Example:

Suppose $Y(s)=\frac{5s+2}{s(5s+4)}$

Then steady state value of Y can be calculated by:

$$Y(\infty)=\lim_{t\to\infty} Y(t)=\lim_{s\to 0}\left[s\frac{(5s+2)}{s(5s+4)}\right]=0.5$$

Multi-Choice Questions

1- What is the Lapace transform of sin t?

a) $\frac{1}{s^2+1}$ b) $\frac{s}{1+s^2}$ c) $\frac{1}{s^2-1}$ d) $\frac{s}{s^2-1}$

2- The transfer function of the first order is:

a) $\frac{1}{Ts+1}$ b) $\frac{1}{Ts}$ c) $\frac{s}{Ts+1}$ d) none of these