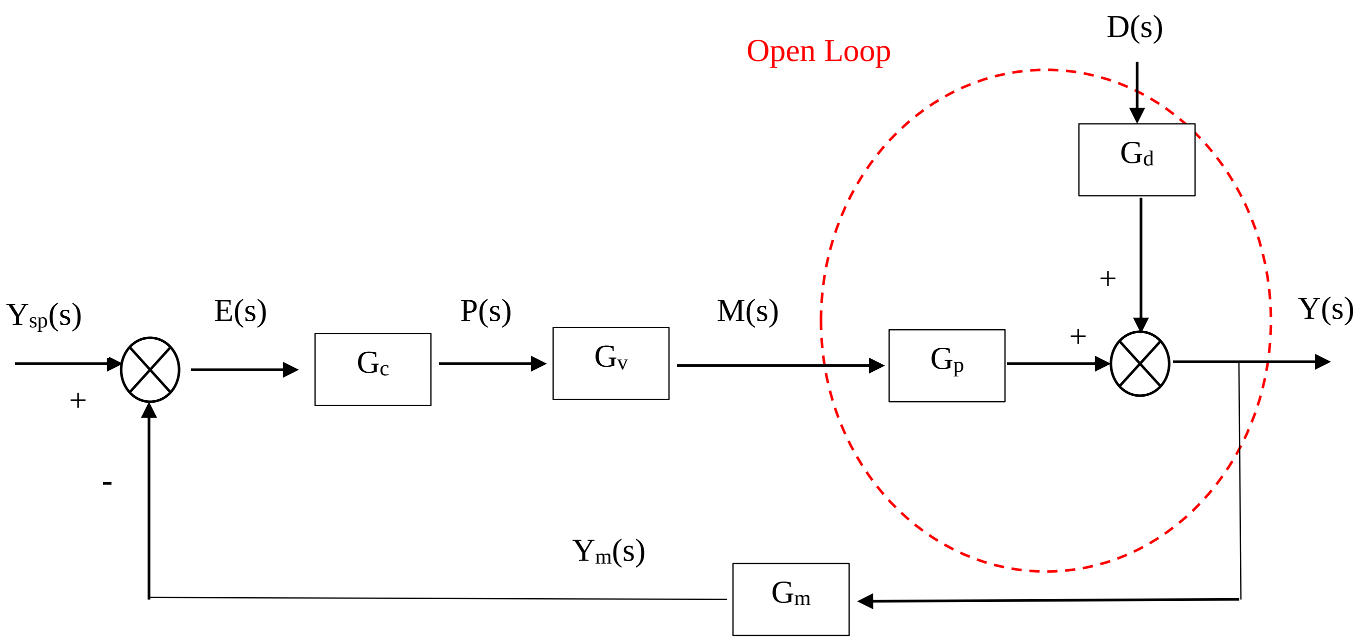
Chapter Three: Transient Response of Simple Control Systems

Chapters Eleven and Twelve in the textbook

In this chapter, results of all previous chapters will be applied to determining the transient response of the simple control.



Closed Loop

Where,

Y_{sp}= Setpoint or desired value

E = Error

M = Manipulated variable

D = Load variable or disturbance

Y = Measured variable

Y_m = Variable produced by measuring device

Process (Open-loop): $$Y(s) = G_p M(s) + G_d D(s) \quad (1)$$

Measuring Device: $$Y_m(s) = G_m Y(s) \quad (2)$$

Controller System:

Comparator $$E(s) = Y_{sp}(s) - Y_m(s) \quad (3)$$

Controller $$P(s) = G_c E(s) \quad (4)$$

Final Control Element: $$M(s) = G_v P(s) \quad (5)$$

Substitute Equation 5 into 1,

$$Y(s) = G_p G_v P(s) + G_d D(s) \quad (6)$$

Substitute Equation 4 into 6,

$$Y(s) = G_p G_v G_c E(s) + G_d D(s) \quad (7)$$

Subsitute Equation 2 into 8,

$$Y(s) = G_p G_v G_c \left(Y_{sp}(s) - Y_m(s)\right) + G_d D(s) \quad (8)$$

Subsitute Equation 3 into 7,

$$Y(s) = G_p G_v G_c \left(Y_{sp}(s) - G_m Y(s)\right) + G_d D(s) \quad (9)$$

Rearrange Equation 9,

$$Y(s) = G_p G_v G_c Y_{sp}(s) - G_p G_v G_c G_m Y(s) + G_d D(s) \quad (10)$$

$$Y(s) + G_p G_v G_c G_m Y(s) = G_p G_v G_c Y_{sp}(s) + G_d D(s) \quad (11)$$

$$\left(1 + G_p G_v G_c G_m\right) Y(s) = G_p G_v G_c Y_{sp}(s) + G_d D(s) \quad (12)$$

The Closed-Loop Transfer Function or Overall Transfer Function

$$Y(s) = \frac{G_p G_v G_c}{1 + G_p G_v G_c G_m} Y_{sp}(s) + \frac{G_d}{1 + G_p G_v G_c G_m} D(s)$$

$$G = G_{forward} = G_p G_v G_c$$

$$Y(s) = \frac{G}{1 + G G_m} Y_{sp}(s) + \frac{G_d}{1 + G G_m} D(s)$$

Closed-loop formula (negative feedback): $\frac{Y}{X} = \frac{G_{forward}}{1+G_{loop}}$

Closed-loop formula (positive feedback): $\frac{Y}{X} = \frac{G_{forward}}{1-G_{loop}}$

Where,

$G_{forward} = G_p G_v G_c$ = The product of transfer functions in the forward path between locations of X and Y.

$G_{loop} = G_p G_v G_c G_m$ = The product of all transfer functions in the loop.

Overall Transfer Function for Change in Set Point

The response to a change in the set point $Y_{sp}(s)$ or desired value, obtained by setting $D(s) = 0$, represents the solution to the servo problem. So, the overall transfer function is as follows:

$$Y(s) = \frac{G_p G_v G_c}{1 + G_p G_v G_c G_m} Y_{sp}(s) \quad \text{or} \quad Y(s) = \frac{G}{1 + GG_m} Y_{sp}(s)$$

The feedback controllers act in such a way as to keep Y close to the changing Y_{sp}.

Overall Transfer Function for Change in Load

The response to a change in the load $D(s)$ or disturbance, obtained by setting $Y_{sp}(s) = 0$, represents the solution to the regulatory problem. So, the overall transfer function is as follows:

$$Y(s) = \frac{G_d}{1 + G_p G_v G_c G_m} D(s) \quad \text{or} \quad Y(s) = \frac{G_d}{1 + GG_m} D(s)$$

The feedback controller tries to eliminate an impact of the load change D to keep Y at the desired setpoint.

Effect of controllers on the transient response of simple control systems

1. **Proportional effect**

The closed-loop transfer function is:

$$Y(s) = \frac{G_p G_v G_c}{1 + G_p G_v G_c G_m} Y_{sp}(s) + \frac{G_d}{1 + G_p G_v G_c G_m} D(s)$$

Consider, $G_m = G_v = 1$

$G_c = K_c$ P control

$G_p = \frac{K_p}{\tau_p s+1}$ First-order equation

$G_d = \frac{K_d}{\tau_p s+1}$

Substitute in the equation above:

$$Y(s) = \frac{K_c\left(\frac{K_p}{\tau_p s + 1}\right)}{1 + K_c\left(\frac{K_p}{\tau_p s + 1}\right)} Y_{sp}(s) + \frac{\frac{K_d}{\tau_p s + 1}}{1 + K_c\left(\frac{K_p}{\tau_p s + 1}\right)} D(s)$$

$$Y(s) = \frac{K_c\left(\frac{K_p}{\tau_p s + 1}\right)}{\frac{\tau_p s + 1}{\tau_p s + 1} + K_c\left(\frac{K_p}{\tau_p s + 1}\right)} Y_{sp}(s) + \frac{\frac{K_d}{\tau_p s + 1}}{\frac{\tau_p s + 1}{\tau_p s + 1} + K_c\left(\frac{K_p}{\tau_p s + 1}\right)} D(s)$$

Rearrange,

$$Y(s) = \frac{K_c K_p}{\tau_p s + 1 + K_c K_p} Y_{sp}(s) + \frac{K_d}{\tau_p s + 1 + K_c K_p} D(s)$$

Divide the equation above by $1 + K_c K_p$

$$Y(s) = \frac{\frac{K_c K_p}{1 + K_c K_p}}{\frac{\tau_p}{1 + K_c K_p} s + 1} Y_{sp}(s) + \frac{\frac{K_d}{1 + K_c K_p}}{\frac{\tau_p}{1 + K_c K_p} s + 1} D(s)$$

$$\mathrm{Y(s)} = \frac{A_1}{\tau_1 \mathrm{s} + 1} \mathrm{Y_{sp}(s)} + \frac{A_2}{\tau_1 \mathrm{s} + 1} \mathrm{D(s)}$$

Where,

$$A_1 = \frac{\mathrm{K_c K_p}}{1+\mathrm{K_c K_p}} \qquad \tau_1 = \frac{\tau_\mathrm{p}}{1+\mathrm{K_c K_p}} \qquad A_2 = \frac{\mathrm{K_d}}{1+\mathrm{K_c K_p}}$$

The close-loop response has the following characteristics:

1. It remains first order with respect to load and set point changes.
2. The time constant has been reduced ($\tau_1 < \tau_p$) which mean that the closed-loop response has become faster than the open loop response, to change in set point or load.
3. The static gain has been decreased.

Offset

The difference (at steady state) between the desired value of the controlled variable (set point) and the actual value of the controlled variable. The offset is actually the steady-state value of the error (for the case of unity feedback) as follows:

$$\text{Offset} = \text{New set point} - \text{Ultimate mesured value}$$

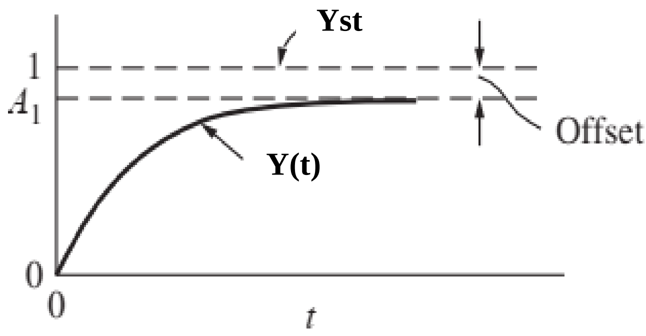
$$\text{Ultmate mesured value} = \lim_{\mathrm{s}\to 0} \mathrm{s\,Y(s)} = \lim_{\mathrm{t}\to\infty} \mathrm{y(t)}$$

For the servo problem, $\mathrm{D(s)} = 0$

$$\mathrm{Y(s)} = \frac{A_1}{\tau_1 \mathrm{s} + 1} \mathrm{Y_{sp}(s)}$$

$$\text{Offset} = 1 - \lim_{\mathrm{s}\to 0} \mathrm{s\,Y(s)} \to 1 - \mathrm{s}\frac{A_1}{\tau_1 \mathrm{s} + 1} \mathrm{Y_{sp}(s)} \to 1 - \mathrm{s}\frac{A_1}{\tau_1 \mathrm{s} + 1}\left(\frac{1}{s}\right) \to 1 - A_1$$

$$\text{Offset} = 1 - \frac{\mathrm{K_c K_p}}{1 + \mathrm{K_c K_p}} \to \frac{1}{1 + \mathrm{K_c K_p}}$$



The offset decreases as K_c increases, and in theory, the offset could be made as small as desired by increasing K_c to a sufficiently large value

$$K_c \uparrow \quad \text{offset} \downarrow$$

For the regulator problem, $Y_{sp}(s) = 0$

$$Y(s) = \frac{A_2}{\tau_1 s + 1} D(s)$$

$$\text{Offset} = 0 - \lim_{s \to 0} s\,Y(s) \to 0 - s\frac{A_2}{\tau_1 s + 1} D(s) \to 0 - s\frac{A_1}{\tau_1 s + 1}\left(\frac{1}{s}\right) \to -A_2$$

$$\text{Offset} = -\frac{K_d}{1 + K_c K_p}$$

As for the case of a step change in set point, the absolute value of the offset is reduced as a controller gain K_c is increased.

2. Integral effect

The closed-loop transfer function is:

$$Y(s) = \frac{G_p G_v G_c}{1 + G_p G_v G_c G_m} Y_{sp}(s) + \frac{G_d}{1 + G_p G_v G_c G_m} D(s)$$

Consider, $G_m = G_v = 1$

$G_c = \frac{K_c}{\tau_I s}$ I control

$G_p = \frac{K_p}{\tau_p s+1}$ First-order equation

$G_d = \frac{K_d}{\tau_p s+1}$

Substitute in the equation above:

$$Y(s) = \frac{\frac{K_c}{\tau_I s}\left(\frac{K_p}{\tau_p s + 1}\right)}{1 + \frac{K_c}{\tau_I s}\left(\frac{K_p}{\tau_p s + 1}\right)} Y_{sp}(s) + \frac{\frac{K_d}{\tau_p s + 1}}{1 + \frac{K_c}{\tau_I s}\left(\frac{K_p}{\tau_p s + 1}\right)} D(s)$$

$$Y(s) = \frac{\frac{K_c K_p}{\tau_I s(\tau_p s + 1)}}{\frac{\tau_I s(\tau_p s + 1)}{\tau_I s(\tau_p s + 1)} + \frac{K_c K_p}{\tau_I s(\tau_p s + 1)}} Y_{sp}(s) + \frac{\frac{K_d}{\tau_p s + 1}}{\frac{\tau_I s(\tau_p s + 1)}{\tau_I s(\tau_p s + 1)} + \frac{K_c K_p \tau_I s + K_c K_p}{\tau_I s(\tau_p s + 1)}} D(s)$$

$$Y(s) = \frac{K_c K_p}{\tau_I \tau_p s^2 + \tau_I s + K_c K_p} Y_{sp}(s) + \frac{K_d \tau_I s}{\tau_I \tau_p s^2 + \tau_I s + K_c K_p} D(s)$$

$$Y(s) = \frac{K_c K_p}{\tau_I \tau_p s^2 + \tau_I s + K_c K_p} Y_{sp}(s) + \frac{K_d \tau_I s}{\tau_I \tau_p s^2 + \tau_I s + K_c K_p} D(s)$$

Divide the equation above by $K_c K_p$

$$Y(s) = \frac{1}{\frac{\tau_I \tau_p}{K_c K_p} s^2 + \frac{\tau_I}{K_c K_p} s + 1} Y_{sp}(s) + \frac{K_d \tau_I s}{\frac{\tau_I \tau_p}{K_c K_p} s^2 + \frac{\tau_I}{K_c K_p} s + 1} D(s)$$

$$Y(s) = \frac{1}{\tau^2 s^2 + 2\zeta\tau s + 1} Y_{sp}(s) + \frac{K_d \tau_I s}{\tau^2 s^2 + 2\zeta\tau s + 1} D(s)$$

Where,

$$\tau^2 = \frac{\tau_I \tau_p}{K_c K_p} \rightarrow \tau = \sqrt{\frac{\tau_I \tau_p}{K_c K_p}}$$

$$2\varphi\tau = \frac{\tau_I}{K_c K_p} \rightarrow \zeta = \frac{1}{2}\sqrt{\frac{\tau_I}{\tau_p K_c K_p}}$$

For the servo problem, $D(s) = 0$

$$Y(s) = \frac{1}{\tau^2 s^2 + 2\zeta\tau s + 1} Y_{sp}(s)$$

$$Y_{sp}(s) = \frac{1}{s}$$

$$\text{Offset } = 1 - \lim_{s \to 0} s \frac{1}{\tau^2 s^2 + 2\zeta\tau s + 1} \frac{1}{s} \rightarrow 1 - 1 = 0$$

For the regulator problem, $Y_{sp}(s) = 0$

$$Y(s) = \frac{K_d \tau_I s}{\tau^2 s^2 + 2\zeta\tau s + 1} D(s)$$

$$D(s) = \frac{1}{s}$$

$$\text{Offset } = 0 - \lim_{s \to 0} s \frac{K_d \tau_I s}{\tau^2 s^2 + 2\zeta\tau s + 1} \frac{1}{s} \rightarrow 0 - 0 = 0$$

Offset is always zero because of the integral effect.

3. Derivative effect

The closed-loop transfer function is:

$$Y(s) = \frac{G_p G_v G_c}{1 + G_p G_v G_c G_m} Y_{sp}(s) + \frac{G_d}{1 + G_p G_v G_c G_m} D(s)$$

Consider, $G_m = G_v = 1$

$G_c = K_c\tau_d s$ D control

$G_p = \frac{K_p}{\tau_p s+1}$ First-order equation

$G_d = \frac{K_d}{\tau_p s+1}$

Substitute in the equation above:

$$Y(s) = \frac{K_c\tau_d s\left(\frac{K_p}{\tau_p s + 1}\right)}{1 + K_c\tau_d s\left(\frac{K_p}{\tau_p s + 1}\right)} Y_{sp}(s) + \frac{\frac{K_d}{\tau_p s + 1}}{1 + K_c\tau_d s\left(\frac{K_p}{\tau_p s + 1}\right)} D(s)$$

$$Y(s) = \frac{\frac{K_p K_c \tau_d s}{\tau_p s + 1}}{1 + \frac{K_p K_c \tau_d s}{\tau_p s + 1}} Y_{sp}(s) + \frac{\frac{K_d}{\tau_p s + 1}}{1 + \frac{K_p K_c \tau_d s}{\tau_p s + 1}} D(s)$$

$$Y(s) = \frac{\frac{K_p K_c \tau_d s}{\tau_p s + 1}}{\frac{\tau_p s + 1}{\tau_p s + 1} + \frac{K_p K_c \tau_d s}{\tau_p s + 1}} Y_{sp}(s) + \frac{\frac{K_d}{\tau_p s + 1}}{\frac{\tau_p s + 1}{\tau_p s + 1} + \frac{K_p K_c \tau_d s}{\tau_p s + 1}} D(s)$$

$$Y(s) = \frac{K_p K_c \tau_d s}{\tau_p s + 1 + K_p K_c \tau_d s} Y_{sp}(s) + \frac{K_d}{\tau_p s + 1 + K_p K_c \tau_d s} D(s)$$

$$Y(s) = \frac{K_p K_c \tau_d s}{(\tau_p + K_p K_c \tau_d)s + 1} Y_{sp}(s) + \frac{K_d}{(\tau_p + K_p K_c \tau_d)s + 1} D(s)$$

The close-loop response has the following characteristics:

1. The derivative control does not change the order of the reponse.
2. The effective time constant of the closed-loop response ($\tau_p + K_p K_c \tau_d > \tau_p$) . This means that the response of the controlled process is slower than that of the original first-order process and as K_c increase the response become slower.

4. Proportional-Integral effect (PI effect)

Combination of propertional and integral control modes lead to the follwing effects on the response of closed-loop system.

1. The order of the response inceases and the offset is eliminated (I effect).
2. As K_c increses, the response becomes fater (P and I effects) and more oscillatory to set point changes [ovesrshoot and decay ratio increase (I effect)]. A large value of the K_c creates a very sensitive response and may lead to instability.
3. As τ_I decreases, for constant K_c, the reponse becomes faster but more oscillatory with higher overshoot and decay ratio (I effect).

   For fixed value of τ_I: as $K_c \uparrow, \varphi \uparrow \quad Max. deviation \downarrow \quad Oscillations \downarrow$

   For fixed value of K_c: as $\tau_I \downarrow, \varphi \downarrow \quad Max. deviation \downarrow \quad Oscillations \uparrow$

5. Proportional-Integral-Derivative effect (PID effect)

To increase speed of the closed loop response, increase the value of the controller gain K_c. But increasing enough K_c to have an acceptable speed, the response become more oscillatory and may lead to unstability.

The introduction of the derivative mode brings a stability effect to the system. Thus to achive

1. Acceptable response speed by selecting an appropriate value for the gain K_c.
2. Maintaining moderate overshoot and decay ratios.

Problems

12.1. The set point of the control system shown in Fig. P12–1 is given a step change of 0.1 unit. Determine

(*a*) The maximum value of C and the time at which it occurs

(*b*) The offset

(*c*) The period of oscillation

Draw a sketch of $C(t)$ as a function of time.

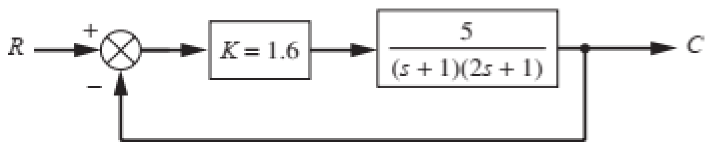


FIGURE P12–1

12.2. The control system shown in Fig. P12–2 contains a PID controller.

(*a*) For the closed loop, develop formulas for the natural period of oscillation τ and the damping factor ζ in terms of the parameters K, τ_D, τ_I, and τ_1.

For the following parts, $\tau_D = \tau_I = 1$ and $\tau_1 = 2$,

(*b*) Calculate ζ when K is 0.5 and when K is 2.

(*c*) Do ζ and τ approach limiting values as K increases, and if so, what are these values?

(*d*) Determine the offset for a unit-step change in load if K is 2.

(*e*) Sketch the response curve (C versus t) for a unit-step change in load when K is 0.5 and when K is 2.

(*f*) In both cases of part (*e*) determine the maximum value of C and the time at which it occurs.

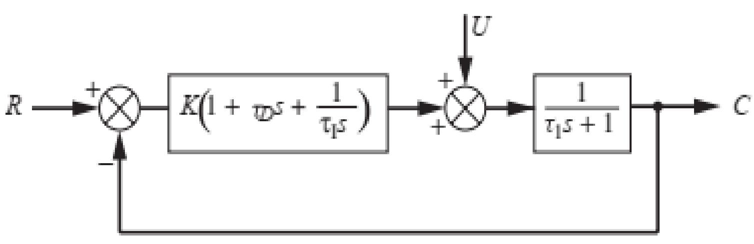


FIGURE P12–2

12.3. The location of a load change in a control loop may affect the system response. In the block diagram shown in Fig. P12–3, a unit-step change in load enters at either location 1 or location 2.

(*a*) What is the frequency of the transient response when the load enters at location 1 and when the load enters at location 2?

(*b*) What is the offset when the load enters at location 1 and when it enters at location 2?

(*c*) Sketch the transient response to a step change in U_1 and to a step change in U_2.

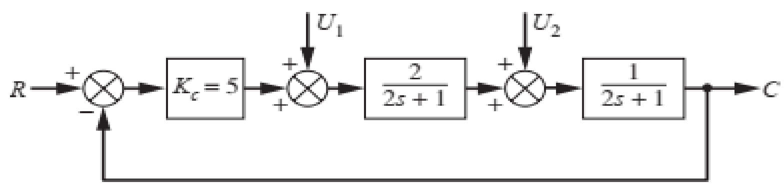


FIGURE P12–3

12.4. Consider the liquid-level control system shown in Fig. P12–4. The tanks are noninteracting. The following information is known:

- The resistances on the tanks are linear. These resistances were tested separately, and it was found that if the steady-state flow rate q cfm is plotted against steady-state tank level h ft, the slope of the line dq/dh is 2 ft^2/min.
- The cross-sectional area of each tank is 2 ft^2.
- The control valve was tested separately, and it was found that a change of 1 psi in pressure to the valve produced a change in flow of 0.1 cfm.
- There is no dynamic lag in the valve or the measuring element.

(*a*) Draw a block diagram of this control system, and in each block give the transfer function, with numerical values of the parameters.

(*b*) Determine the controller gain K_c for a critically damped response.

(*c*) If the tanks were connected so that they were interacting, what is the value of K_c needed for critical damping?

(*d*) Using 1.5 times the value of K_c determined in part (*c*), determine the response of the level in tank 2 to a step change in set point of 1 in of level.

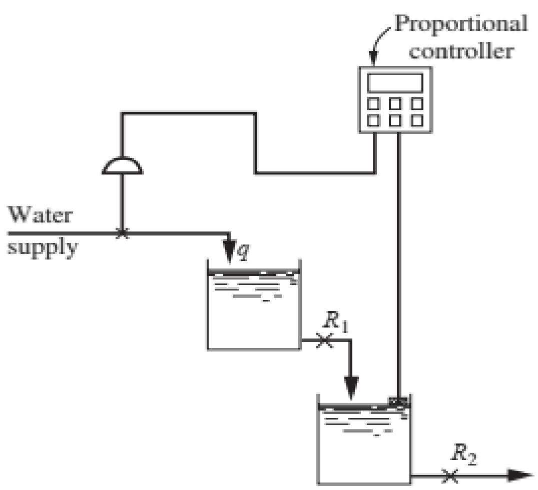


FIGURE P12–4

12.5. A PD controller is used in a control system having a first-order process and a measurement lag as shown in Fig. P12–5.

(*a*) Find expressions for ζ and τ for the closed-loop response.

(*b*) If $\tau_1 = 1$ min and $\tau_m = 10$ s, find K_c so that $\zeta = 0.7$ for the two cases (1) $\tau_D = 0$ s and (2) $\tau_D = 3$ s.

(*c*) Compare the offset and period realized for both cases, and comment on the advantage of adding the derivative mode.

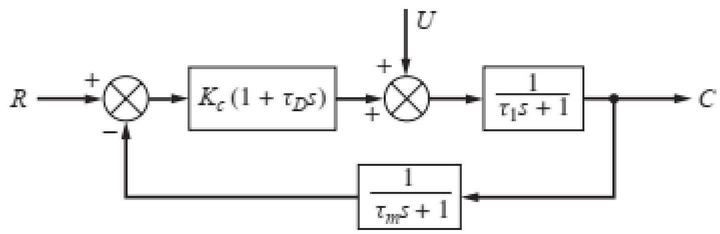


FIGURE P12–5

12.6. The thermal system shown in Fig. P12–6 is controlled by a PD controller. These data are given:

$$w = 250 \text{ lb/min}$$
$$\rho = 62.5 \text{ lb/ft}^3$$
$$V_1 = 4 \text{ ft}^3$$
$$V_2 = 5 \text{ ft}^3$$
$$V_3 = 6 \text{ ft}^3$$
$$C = 1 \text{ Btu/(lb}\cdot{}^\circ\text{F)}$$

A change of 1 psi from the controller changes the flow rate of heat q by 500 Btu/min. The temperature of the inlet stream may vary. There is no lag in the measuring element.

(*a*) Draw a block diagram of the control system with the appropriate transfer function in each block. Each transfer function should contain numerical values of the parameters.

(*b*) From the block diagram, determine the overall transfer function relating the temperature in tank 3 to a change in set point.

(*c*) Find the offset for a unit-step change in inlet temperature if the controller gain K_c is 3 psi/°F of temperature error and the derivative time is 0.5 min.

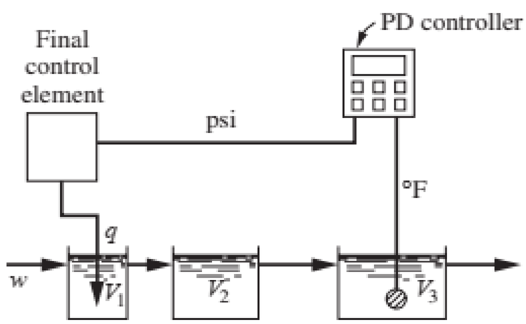


FIGURE P12–6

12.7. (*a*) For the control system shown in Fig. P12–7, obtain the closed-loop transfer function C/U.

(*b*) Find the value of K_c for which the closed-loop response has a ζ of 2.3.

(*c*) Find the offset for a unit-step change in U if $K_c = 4$.

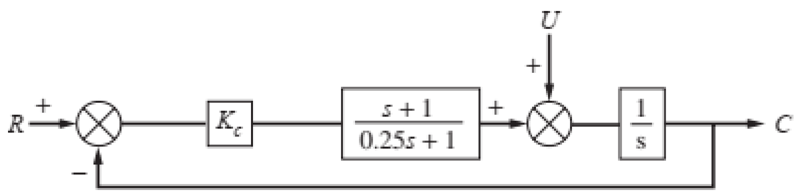


FIGURE P12–7

12.8. For the control system shown in Fig. P12–8, determine

(*a*) $C(s)/R(s)$

(*b*) $C(\infty)$

(*c*) Offset

(*d*) $C(0.5)$

(*e*) Whether the closed-loop response is oscillatory

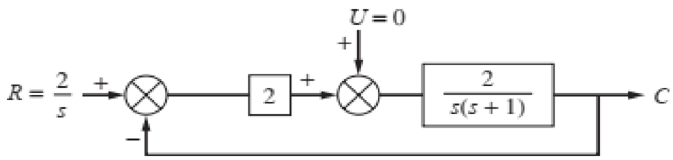


FIGURE P12–8

12.9. For the control system shown in Fig. P12–9, determine an expression for $C(t)$ if a unit-step change occurs in R. Sketch the response C(t) and compute C(2).

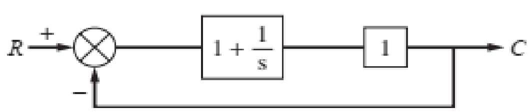


FIGURE P12–9

12.10. Compare the responses to a unit-step change in set point for the system shown in Fig. P12–10 for both negative feedback and positive feedback. Do this for K_c of 0.5 and 1.0. Compare these responses by sketching C(t).

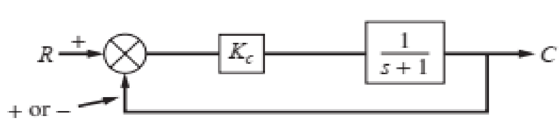


FIGURE P12–10